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Three-dimensional theory of water impact. Part 2. Linearized Wagner problem

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The three-dimensional problem of blunt-body impact onto a free surface of an ideal and incompressible liquid is considered within the Wagner approximation. This approximation is formally valid during an initial stage, when the depth of penetration is small, the wetted part of the body can be approximately replaced with a flat disk and the boundary conditions can be linearized and imposed on the undisturbed liquid surface. In the present context this problem will be referred to as the classical Wagner problem. However the classical Wagner problem of impact is nonlinear despite the fact that the equations of liquid motion and boundary conditions are linearized. The reason is that the contact region between the liquid and the entering body is unknown in advance and has to be determined together with the liquid flow. Several exact solutions of the three-dimensional Wagner problem are known as detailed in Part 1 (*J. Fluid Mech.* vol. 440, 2001, p. 293). Among these solutions the axisymmetric one is the simplest. In this paper, an additional linearization of the Wagner problem is considered. This linearization is performed on the basis of an axisymmetric solution via a perturbation technique. The small parameter ϵ is a measure of the discrepancy of the actual shape with respect to the closest axisymmetric shape. The method of solution of this problem is detailed here. The resulting solutions are compared to available exact solutions. Three shapes are studied: elliptic paraboloid; inclined cone; and pyramid. These shapes must be blunt in the vicinity of the initial contact point and hence only small deadrise angles can be considered. The stability of the obtained solutions is analysed. The second-order solution of the present Wagner problem with respect to ϵ is considered. That yields the leading-order correction to the hydrodynamic force which acts on an almost axisymmetric body entering liquid vertically. Other nonlinearities are not accounted for. Among them, there are the nonlinear terms in the boundary conditions and the actual geometry of the wetted body surface. Both the vertical and the horizontal components of the hydrodynamic force are obtained. For the inclined cone, comparisons with available experimental data are shown. The method developed can be helpful in testing other numerical approaches and optimizing the shape of the entering body accounting for three-dimensional effects. This paper appears as a necessary intermediate step before solving the general three-dimensional classical Wagner problem in Part 3.

1. Introduction

The increasing needs of the offshore industry require further development of accurate numerical methods for solving the three-dimensional impact problem. Euler or Navier–Stokes equation solvers are becoming more and more stable, but their

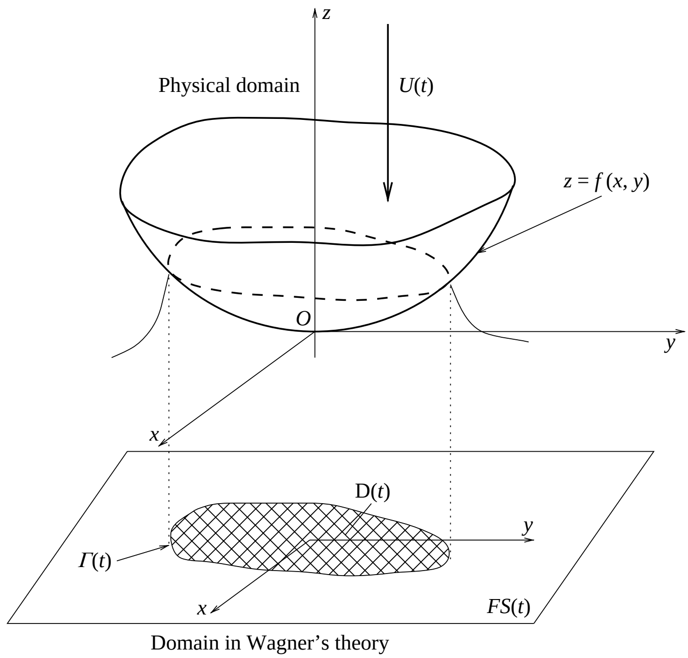

FIGURE 1. Sketches of three-dimensional flow pattern for normal penetration of a blunt body into a liquid within the original problem and within the Wagner approximation: D(t), contact region; FS(t), liquid free surface; $\Gamma(t)$, contact line. In the physical domain the spray jet is not shown.

computating costs still make them relatively unattractive, even though theoretically they can handle arbitrary configurations. As an alternative, the Wagner approach offers the easiest way to solve three-dimensional impact problems. This approach provides predictions that are helpful at the preliminary design stages and yields reference data to validate other numerical methods.

The three-dimensional problem of unsteady liquid flow arising when a blunt body enters an ideal incompressible liquid through the free surface, is considered (figure 1). Initially, the liquid is at rest and occupies the lower half-space, $z<0$. At the initial instant of time, $t=0$, the body starts to penetrate the liquid vertically with the body velocity $U(t)$ being prescribed. External mass forces and surface tension are neglected. The liquid flow caused by the impact is assumed to be irrotational. The liquid flow is hence described by the velocity potential $\phi(x, y, z, t)$ which satisfies a mixed boundary-value problem abundantly described in the literature. Within the classical Wagner approximation (Wagner 1932), the wetted part of the body is approximated by the flat disk of the corresponding shape (flat-disk approximation), which is possible for blunt bodies. The elevation of the disturbed free surface $Z(x, y, t)$ is obtained from the linearized kinematic boundary condition and the pressure distribution $p(x, y, z, t)$ from the linearized Bernoulli equation. The vertical component of the hydrodynamic force $F(t)$ acting on the entering body is evaluated by integration of the pressure distribution $p(x, y, 0, t)$ over the contact region D(t).

As described in figure 1, the division of the liquid boundary into a free surface FS(t) and a wetted region D(t) is unknown in advance and must be determined with the help of an additional condition known as the Wagner condition (see Howison, Ockendon & Oliver 2002). This condition implies that the free-surface elevation and the position of the body surface are equal to each other along the boundary of the contact region $\Gamma(t) = \partial D(t)$. The presence of this condition makes the problem nonlinear and difficult to analyse, despite the fact that both the equations of motion and the boundary conditions are linear. More details about the formulation of the Wagner problem and the main assumptions the Wagner theory is based on, can be found in Howison, Ockendon & Wilson (1991) and Scolan & Korobkin (2001).

The difficulties were resolved in both symmetric two-dimensional and axisymmetric cases, where the dimension of the contact region is described by a single function of time, say, $a(t)$ (Korobkin 1996). In these cases, the Wagner condition can be reduced to a single nonlinear algebraic equation with respect to the function $a(t)$. In the asymmetric two-dimensional problem, we require two functions to describe both the dimension and the position of the contact region. A nonlinear system of algebraic equations with respect to these functions was derived by Scolan *et al.* (1999). Assuming the contact region to be elliptic, the corresponding shapes were obtained by Scolan & Korobkin (2001) by an inverse method. The elliptic contact regions are described with the two semi-axes which are both functions of time. This method provides several exact solutions of the Wagner problem.

The classical Wagner approach is known to be applicable to blunt bodies which have a deadrise angle of less than 20°. To deal with larger deadrise angles, the so-called generalized Wagner approach has been developed. However, this generalized approach requires substantial numerical efforts (see Zhao, Faltinsen & Aarnes 1996; Faltinsen & Zhao 1997; or Fabula 1957). To the best of our knowledge, the numerical solution of the generalized Wagner problem – by the boundary-element method for example – do not exist for arbitrary three-dimensional geometries.

In the truly three-dimensional case, the classical Wagner approaches developed so far do not work and the nonlinearity of the problem is still a major obstacle. For an arbitrary shape of the entering body, the contact region is bounded by a two-dimensional curve $\Gamma(t)$ which is referred to as the contact line. The contact region expands with time and its boundary $\Gamma(t)$ must be determined at each time instant together with the liquid flow. To circumvent this difficulty, we suggest linearizing this problem on the basis of a known solution and studying the linearized problem first.

The body shape is described with the Cartesian coordinate system $Oxyz$ where the origin is the initial contact point at the initial instant. The function $f(x, y)$ describes the body shape with $f(0, 0) = 0$ and the gradient $|\nabla f|$ being much smaller than unity close to the impact point. The linearization is performed about an axisymmetric solution. This implies that the shape function $f(x, y)$ can be decomposed in polar coordinates as

$$f(x, y) = f_0(r) + \epsilon F(r, \theta), \tag{1}$$

where $x = r\cos\theta$, $y = r\sin\theta$, $\epsilon \ll 1$, the functions $f_0(r)$ and $F(r, \theta)$ are smooth and $F(r, \theta) = O(1)$, $f_0(r) = O(1)$, $F(0, \theta) = 0$, $f_0(0) = 0$, $f_0'(r) = \mathrm{d}f_0/\mathrm{d}r \ll 1$ in a small vicinity of the impact point $r = 0$. The function $F(r, \theta)$ can also be dependent on the small parameter ϵ. Physically, ϵ is a measure of the discrepancy between the actual shape and the axisymmetrical one (see figure 2 described later in the text).

The positive non-dimensional parameter ϵ is considered here as the parameter of linearization. For example, in the case of an elliptic paraboloid, $f(x, y) = x^2/(2r_x) + y^2/(2r_y)$, $r_y > r_x$, decomposition (1) provides $f(x, y) = f_0(r) + \epsilon f_0(r)\cos 2\theta$, where $f_0(r) = r^2(r_x^{-1} + r_y^{-1})/4$ and $\epsilon = (r_y - r_x)/(r_y + r_x)$. If $r_y = 50$ cm and $r_x = 37.5$ cm, then $\epsilon = 1/7$. We shall determine the asymptotic solution of the Wagner problem as $\epsilon \to 0$ for a body, the shape of which is described by (1).

Asymptotic analysis of the Wagner problem is not simple because both the flow and the geometry of the contact region depend on the small parameter ϵ. We do not know how to perform the asymptotic analysis of the classical Wagner problem formulated in terms of the velocity potential. In this paper, another approach is investigated. It is based on the regularization of the boundary-value problem for the velocity potential ϕ (see Korobkin 1982). In this approach, we deal with a new function $\varphi(x, y, z, t)$ such that $\varphi_{,t} = \phi$. By definition, this function is the displacement potential. The corresponding boundary-value problem is detailed in Howison *et al.* (1991). In the present study, we show how to reduce this boundary-value problem to an integral equation which is suitable for asymptotic analysis. The analysis is focused on the first-order asymptotic solution. The higher-order solutions are discussed. The second-order solution is also considered, in order to determine the hydrodynamic force on the entering body.

The formulation of the Wagner problem and its regularization are presented in § 2. The asymptotic analysis of the regularized problem is given in § 3. The zeroth-order solution is derived in § 4 and the first-order solution in § 5. Asymptotics of hydrodynamic force on the entering body and the second-order solution are studied in § 6. The asymptotic results for the elliptic paraboloid entry problem are detailed and compared to the available exact solutions in § 7. Sections § 8 and § 9 describe applications of the developed technique to the entry problems of an inclined cone and a pyramid, respectively. The results obtained for the inclined cone are compared to available experimental data. Finally in § 10, the results obtained are summarized and some directions for further work are outlined.

2. Formulation of the Wagner problem

Within the Wagner approximation, the liquid flow caused by the entering body is described by the velocity potential $\phi(x, y, z, t)$, for which the boundary-value problem has the form

$$\left.\begin{array}{ll} \phi_{,xx} + \phi_{,yy} + \phi_{,zz} = 0, & z < 0, \\ \phi = 0, & z = 0,\ (x, y) \in \mathrm{FS}(t), \\ \phi_{,z} = -U(t), & z = 0,\ (x, y) \in \mathrm{D}(t), \\ \phi \to 0, & (x^2 + y^2 + z^2) \to \infty, \end{array}\right\} \tag{2}$$

where the regions FS(t) and D(t) are parts of the plane $z = 0$ and correspond to the free surface and the wetted area of the body, respectively. A closed curve, which separates the regions FS(t) and D(t), is denoted $\Gamma(t)$ and is referred to as the contact line. The shape of the disturbed free surface is described by the equation $z = Z(x, y, t)$, where $(x, y) \in \mathrm{FS}(t)$. The function $Z(x, y, t)$ is calculated using the linearized kinematic boundary condition

$$Z_{,t}(x, y, t) = \phi_{,z}(x, y, 0, t) \qquad (z = 0,\ (x, y) \in \mathrm{FS}(t)), \tag{3}$$

$$Z(x, y, 0) = 0. \tag{4}$$

The position of the contact line $\Gamma(t)$ is determined with the help of the additional condition

$$f(x, y) - h(t) = Z(x, y, t) \qquad ((x, y) \in \Gamma(t)), \tag{5}$$

which is known as the Wagner condition (Wagner 1932). The pressure in the liquid is given by the linearized Bernoulli equation $p(x, y, z, t) = -\rho_0 \phi_{,t}(x, y, z, t)$, where ρ_0 is the liquid density. The vertical component of the hydrodynamic force $F(t)$ on the entering body follows from the pressure integration over the wetted area D(t)

$$F(t) = -\rho_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{D(t)} \phi(x, y, 0, t)\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

The inverse method to obtain solutions of the Wagner problem (2)–(5) was presented by Scolan & Korobkin (2001). With this method, both the contact line $\Gamma(t)$ and the body velocity $U(t)$ are prescribed and the corresponding shape of the entering body is determined from equation (5). However, if the impact conditions–shape function $f(x, y)$ and the body velocity $U(t)$–are given, other approaches must be developed.

The approach based on regularization of the Wagner problem (2)–(5) is considered in the present paper. We know that the velocity potential ϕ on the liquid boundary, $z = 0$, is continuous and the liquid velocity $\nabla\phi$ is square integrable. The components of the velocity vector are singular at the moving contact line $\Gamma(t)$. In order to regularize the problem, the displacement potential

$$\varphi(x, y, z, t) = \int_0^t \phi(x, y, z, \tau)\,\mathrm{d}\tau \tag{7}$$

is introduced. The boundary-value problem for the displacement potential is obtained by integrating equations (2) and (3) in time, with the Wagner condition (5) taken into account. We find (see Howison *et al.* 1991 for details)

$$\left.\begin{array}{ll} \varphi_{,xx} + \varphi_{,yy} + \varphi_{,zz} = 0, & z < 0, \\ \varphi = 0, & z = 0,\ (x, y) \in \mathrm{FS}(t), \\ \varphi_{,z} = f(x, y) - h(t), & z = 0,\ (x, y) \in \mathrm{D}(t), \\ \varphi \to 0, & (x^2 + y^2 + z^2) \to \infty, \end{array}\right\} \tag{8}$$

The time t plays the role of a parameter in (8). This means that the displacement potential can be found at each time instant, independently of the previous history of the process.

It is worth noting that the integration in equation (7) increases the smoothness of the potential, not only with respect to time t, but also with respect to the spatial variables x, y and z. This implies that the displacement potential is continuously differentiable over the flow region $z \leqslant 0$ and its second derivatives with respect to the spatial variables are square integrable. Taking into account the boundary condition on the liquid free surface, we find that $\varphi(x, y, 0, t) = 0$, $\varphi_{,x}(x, y, 0, t) = 0$ and $\varphi_{,y}(x, y, 0, t) = 0$ along the contact line.

The boundary-value problem (8) leads to the integral equation

$$\frac{1}{2\pi} \iint_{\mathrm{D}(t)} \frac{s(x_0, y_0, t)\,\mathrm{d}x_0\,\mathrm{d}y_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = h(t) - f(x, y) \qquad ((x, y) \in \mathrm{D}(t)), \tag{9}$$

where $s(x, y, t) = \Delta_2\varphi = \varphi_{,xx}(x, y, 0, t) + \varphi_{,yy}(x, y, 0, t)$. The function $s(x, y, t)$ is square integrable. Once the integral equation (9) has been solved, the displacement potential $\tilde{\varphi}(x, y, t) = \varphi(x, y, 0, t)$ in the contact region D(t) is calculated from the solution of the following boundary-value problem for the Poisson equation

$$\left.\begin{array}{ll} \Delta_2\tilde{\varphi} = s(x, y, t), & (x, y) \in \mathrm{D}(t), \\ \tilde{\varphi} = 0, & (x, y) \in \Gamma(t). \end{array}\right\} \tag{10}$$

Moreover, the continuity of the displacements at the contact line leads to the equation

$$\frac{\partial\tilde{\varphi}}{\partial n} = 0 \qquad ((x, y) \in \Gamma(t)), \tag{11}$$

which is used to determine the position of the contact line $\Gamma(t)$. In (11), $\partial\tilde{\varphi}/\partial n$ is

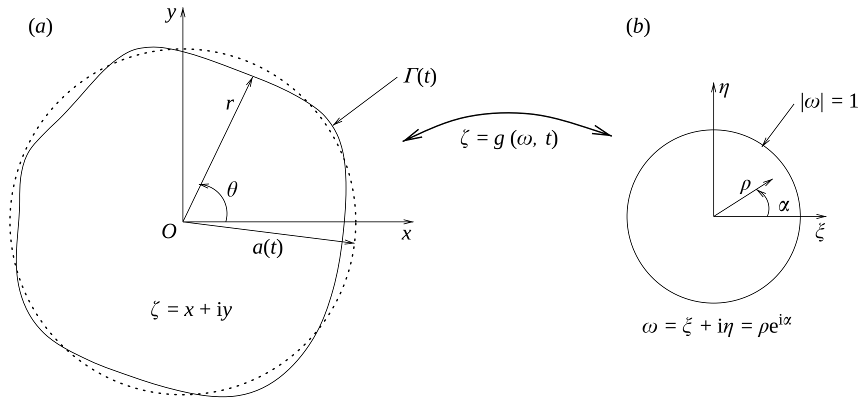

FIGURE 2. Description of the contact line $\Gamma(t)$ at time t ((*a*), solid line) considered as a curve closed to a circle with radius $a(t)$ ((*a*), dotted line). The function g provides conformal mapping of the wetted part of the body at time t (*b*). The physical plane is described with the complex variable $\zeta = x + \mathrm{i}y$. The transformed plane (in (*b*) where the wetted surface is a unit circle) is described with the complex variable $\omega = \xi + \mathrm{i}\eta$.

the normal derivative of the function $\tilde{\varphi}(x, y, t)$, $(x, y) \in \mathrm{D}(t)$, to the contact line. The boundary-value problem (10)–(11) belongs to a wide class of free-boundary problems (see Howison, Morgan & Ockendon 1997) because the boundary $\Gamma(t)$ of the domain $\mathrm{D}(t)$ must be determined together with the function $\tilde{\varphi}(x, y, t)$.

Once problem (9)–(11) has been solved, the free-surface elevation $Z(x, y, t)$ is given as

$$Z(x, y, t) = -\frac{1}{2\pi} \iint_{\mathrm{D}(t)} \frac{s(x_0, y_0, t)\, \mathrm{d}x_0\, \mathrm{d}y_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \qquad ((x, y) \in \mathrm{FS}(t)). \tag{12}$$

Equation (12) follows from (8) and (3). Equations (6), (7), (10) and (11) provide

$$F(t) = -\rho_0 \frac{\mathrm{d}^2}{\mathrm{d}t^2} \iint_{\mathrm{D}(t)} \tilde{\varphi}(x, y, t)\, \mathrm{d}x\, \mathrm{d}y. \tag{13}$$

for wetted surface $\mathrm{D}(t)$ with smooth contour. Far from the contact region, $x^2+y^2 \to \infty$, the free-surface elevation

$$Z(x, y, t) = -\frac{1}{2\pi} \frac{N(t)}{r^3} + O(r^{-2}), \qquad N(t) = \iint_{\mathrm{D}(t)} \tilde{\varphi}(x_0, y_0, t)\, \mathrm{d}x_0\, \mathrm{d}y_0, \tag{14}$$

does not depend on the polar angle to leading order. The obtained asymptotics of the free surface elevation and equation (13) show the importance of the integral of the displacement potential over the contact region denoted as $N(t)$.

The case of an almost axisymmetric entering body, the shape of which is described by equation (1) with a small parameter ϵ, is considered below. It is important to notice that the region $\mathrm{D}(t)$, where the functions $s(x, y, t)$ and $\tilde{\varphi}(x, y, t)$ must be determined, is also dependent on the small parameter ϵ. Therefore, asymptotic methods cannot be applied directly to equations (9)–(11).

To avoid this difficulty, the following parameterization of the contact region $\mathrm{D}(t)$ is introduced. The conformal mapping $\zeta = g(\omega, t)$ is used, where $\zeta = x + \mathrm{i}y$, $\omega = \xi + \mathrm{i}\eta$, $|\omega| \leqslant 1$, $g(0, t) = 0$, $g(\omega, t) = g_1(\xi, \eta, t) + \mathrm{i}g_2(\xi, \eta, t)$ and the imaginary part of the derivative $\mathrm{Im}[(\partial g/\partial \omega)(0, t)]$ at the centre of the circle is equal to zero. The function $g(\omega, t)$ maps the unit circular disk $|\omega| \leqslant 1$ onto the wetted region $\mathrm{D}(t)$ bounded with $\Gamma(t)$ (figure 2). This function must be determined, together with the displacement potential from equations (9)–(11). In particular, in the axisymmetric case, we have $g(\omega, t) = a(t)\omega$, where the function $a(t)$ is unknown in advance.

In the new variables ξ, η, equations (9)–(11) take the forms

$$\frac{1}{2\pi}\iint_{|\omega_0|<1} \frac{S(\xi_0, \eta_0, t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|g(\omega, t) - g(\omega_0, t)|} = Y(\xi, \eta, t) \qquad (\xi^2 + \eta^2 < 1), \tag{15}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\alpha^2} = S(\rho, \alpha, t) \qquad (\rho < 1), \tag{16}$$

$$\Phi = 0 \qquad (\rho = 1), \tag{17}$$

$$\frac{\partial\Phi}{\partial\rho} = 0 \qquad (\rho = 1), \tag{18}$$

where

$$\left.\begin{aligned} S(\xi, \eta, t) &= s[g_1(\xi, \eta, t), g_2(\xi, \eta, t), t]|(\partial g/\partial\omega)(\omega, t)|^2, \\ Y(\xi, \eta, t) &= h(t) - f[g_1(\xi, \eta, t), g_2(\xi, \eta, t)], \\ \Phi(\rho, \alpha, t) &= \tilde{\varphi}[g_1(\xi, \eta, t), g_2(\xi, \eta, t), t], \quad \xi = \rho\cos\alpha, \quad \eta = \rho\sin\alpha. \end{aligned}\right\} \tag{19}$$

The hydrodynamic force on the entering body $F(t)$ and the free-surface elevation $Z(x, y, t)$ are given in the new variables as

$$F(t) = -\rho_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}\iint_{\rho<1} \Phi(\rho, \alpha, t)|(\partial g/\partial\omega)(\omega, t)|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha, \tag{20}$$

$$Z(x, y, t) = -\frac{1}{2\pi}\iint_{|\omega_0|<1} \frac{S(\xi_0, \eta_0, t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\zeta - g(\omega_0, t)|} \qquad (\zeta = x + \mathrm{i}y, (x, y) \in \mathrm{FS}(t)). \tag{21}$$

The integrals in both (20) and (21) are evaluated after the function $\Phi(\rho, \alpha, t)$ has been obtained.

The Wagner problem can now be formulated as follows: it is necessary to determine the function $\Phi(\rho, \alpha, t)$ and the conformal mapping $g(\omega, t)$, which satisfy equations (15)–(18) in the unit circle. It is important to note that in such a formulation, the domain, where the solution has to be found, does not depend either on time or on the body shape. In this formulation the Wagner problem is suitable for its asymptotic analysis.

3. Asymptotic analysis of the Wagner problem

In the case of an almost axisymmetric body entering a liquid, the shape function $f(x, y)$ is given by equation (1), where ϵ is a small parameter. An approximate solution of equations (15)–(18) as $\epsilon \to 0$ is sought in the form

$$\left.\begin{aligned} \Phi(\rho, \alpha, t) &= \Phi_0(\rho, t) + \epsilon\Phi_1(\rho, \alpha, t) + \epsilon^2\Phi_2(\rho, \alpha, t) + O(\epsilon^3), \\ S(\rho, \alpha, t) &= S_0(\rho, t) + \epsilon S_1(\rho, \alpha, t) + \epsilon^2 S_2(\rho, \alpha, t) + O(\epsilon^3), \\ g(\omega, t) &= a(t)[\omega + \epsilon H(\omega, t, \epsilon)], \\ H(\omega, t, \epsilon) &= H_0(\omega, t) + \epsilon H_1(\omega, t) + O(\epsilon^2), \end{aligned}\right\} \tag{22}$$

with the leading-order terms reflecting the fact that the solution is axisymmetric for

$\epsilon = 0$. The following equalities are then used

$$\left.\begin{aligned} &r = |g(\omega, t)|, \qquad \theta = \arg[g(\omega, t)] = \alpha + \arg[1 + \epsilon\omega^{-1}H(\omega, t, \epsilon)], \\ &|g(\omega, t)| = a(t)|\omega|\sqrt{1 + 2\epsilon|\omega|^{-2}\,\mathrm{Re}\,(\bar{\omega}H) + \epsilon^2|\omega|^{-2}|H|^2}, \\ &\frac{1}{|g(\omega, t) - g(\omega_0, t)|} = \frac{1}{a(t)|\omega - \omega_0|}[1 + 2\epsilon\mathrm{Re}(T) + \epsilon^2|T|^2]^{-1/2}, \\ &[1 + 2\epsilon\mathrm{Re}(T) + \epsilon^2|T|^2]^{-1/2} = \sum_{n=0}^{\infty}(-1)^n\epsilon^n|T|^n P_n(\cos\Psi), \end{aligned}\right\} \tag{23}$$

with

$$T(\omega, \omega_0, t, \epsilon) = \frac{H(\omega, t, \epsilon) - H(\omega_0, t, \epsilon)}{\omega - \omega_0} = |T|\mathrm{e}^{\mathrm{i}\Psi}, \tag{24}$$

where $P_n(x)$ are the Legendre polynomials and the complex conjugate is denoted $\bar{\omega} = \xi - \mathrm{i}\eta$. Equalities (23) and (24) lead to the asymptotic formulae

$$|g(\omega, t)| = a(t)\rho + \epsilon\frac{a(t)}{\rho}\mathrm{Re}(\bar{\omega}H_0) + O(\epsilon^2), \qquad \arg[g(\omega, t)] = \alpha + O(\epsilon), \tag{25}$$

$$\frac{1}{|g(\omega, t) - g(\omega_0, t)|} = \frac{1}{a(t)|\omega - \omega_0|}[1 - \epsilon\mathrm{Re}(T_0) + O(\epsilon^2)], \tag{26}$$

$$T_0(\omega, \omega_0, t) = \frac{H_0(\omega, t) - H_0(\omega_0, t)}{\omega - \omega_0}. \tag{27}$$

The right-hand side of equation (15) is decomposed as

$$Y(\xi, \eta, t) = Y_0(\xi, \eta, t) + \epsilon Y_1(\xi, \eta, t) + \epsilon^2 Y_2(\xi, \eta, t) + \ldots, \tag{28}$$

where

$$\left.\begin{aligned} &Y_0(\xi, \eta, t) = h(t) - f_0(a\rho), \qquad Y_1(\xi, \eta, t) = -\frac{a(t)}{\rho}f_0'(a\rho)\mathrm{Re}(\bar{\omega}H_0) - F(a\rho, \alpha), \\ &Y_{j+1}(\xi, \eta, t) = -\frac{a(t)}{\rho}f_0'(a\rho)\mathrm{Re}(\bar{\omega}H_j) + \tilde{Y}_{j+1}(\xi, \eta, t). \end{aligned}\right\} \tag{29}$$

The functions $\tilde{Y}_j(\xi, \eta, t)$, where $j \geqslant 1$, are dependent on previous approximations $H_n(\omega, t)$, $0 \leqslant n \leqslant j - 1$.

Substituting the asymptotic formulae (25)–(29) into equations (15)–(18) and collecting the terms of the same order as ϵ^j, $j \geqslant 0$, the successive boundary-value problems are derived for the unknown functions $\Phi_j(\rho, \alpha, t)$, $S_j(\rho, \alpha, t)$ and $H_j(\omega, t)$.

At zeroth order ($j = 0$), we obtain

$$\frac{1}{2\pi}\iint_{|\omega_0|<1}\frac{S_0(\xi_0, \eta_0, t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega - \omega_0|} = a(t)[h(t) - f_0(a\rho)] \qquad (\rho < 1), \tag{30}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_0}{\partial\rho}\right) = S_0(\rho, t) \qquad (\rho < 1), \tag{31}$$

$$\Phi_0 = \frac{\partial\Phi_0}{\partial\rho} = 0 \qquad (\rho = 1). \tag{32}$$

The system of equations (30)–(32) corresponds to the original problem (8) with $\epsilon = 0$.

At first-order ($j = 1$), we obtain

$$\frac{1}{2\pi}\iint_{|\omega_0|<1}\frac{S_1(\xi_0, \eta_0, t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega - \omega_0|} = Q_0(\rho, \alpha, t) \qquad (\rho < 1), \tag{33}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_1}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi_1}{\partial\alpha^2}=S_1(\rho,\alpha,t)\qquad(\rho<1),\tag{34}$$

$$\Phi_1=\frac{\partial\Phi_1}{\partial\rho}=0\qquad(\rho=1),\tag{35}$$

where

$$Q_0(\rho,\alpha,t)=\frac{1}{2\pi}\iint_{|\omega_0|<1}\frac{S_0(\xi_0,\eta_0,t)\mathrm{Re}(T_0)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega-\omega_0|}-\frac{a^2}{\rho}f_0'(a\rho)\mathrm{Re}(\bar{\omega}H_0)-aF(a\rho,\alpha).\tag{36}$$

At higher orders ($j\geqslant 2$), the corresponding equations are similar to (33)–(35), but the right-hand sides of the integral equations become much more complicated. We obtain at the jth order

$$\frac{1}{2\pi}\iint_{|\omega_0|<1}\frac{S_j(\xi_0,\eta_0,t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega-\omega_0|}=\frac{1}{2\pi}\iint_{|\omega_0|<1}\frac{S_0(\xi_0,\eta_0,t)\mathrm{Re}(T_{j-1})\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega-\omega_0|}$$
$$-\frac{a^2}{\rho}f_0'(a\rho)\mathrm{Re}(\bar{\omega}H_{j-1})-aK_j(\rho,\alpha,\mathrm{t})\qquad(\rho<1),\tag{37}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_j}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi_j}{\partial\alpha^2}=S_j(\rho,\alpha,t)\qquad(\rho<1),\tag{38}$$

$$\Phi_j=\frac{\partial\Phi_j}{\partial\rho}=0\qquad(\rho=1).\tag{39}$$

Here, the functions $K_j(\rho,\alpha,t)$ depend on previous approximations: $H_n(\omega,t)$, $0\leqslant n\leqslant j-2$ and $S_m(\rho,\alpha,t)$, $0\leqslant m\leqslant j-1$, and are considered as given. It can be seen that equations (37)–(39) are reduced to equations (33)–(35) if we take $j=1$ and $K_1(\rho,\alpha,t)=F(a\rho,\alpha)$ in the former. Therefore, once the first-order solution has been found, the same procedure can be used to find the solution at any order. In this paper, attention is focused on the first-order solution.

Once the asymptotic solution of equations (15)–(18) has been obtained, we evaluate numerically the approximate shape of the free surface using (21) and derive the asymptotics of the hydrodynamic force from equation (20).

4. Zeroth-order solution

The problem of axisymmetric body impact with the Wagner approximation has been analysed in detail starting with the pioneering paper by Schmieden (1953). As an alternative to Schmieden's analysis, in this paper, the problem is formulated in terms of the displacement potential. The latter formulation makes it possible to derive a simple equation for the radius of the contact region $a(t)$ (see Korobkin 1985) and avoids difficulties with treatment of the Wagner condition. It should be noted that the Wagner condition in its classical formulation leads to a singular and nonlinear integral equation for the function $a(t)$. This integral equation is a source of numerical difficulties.

By taking into account the correspondence between equations (8) and system (9)–(11), it is clear that equations (30)–(32) are formally equivalent to the following boundary-value problem with mixed boundary conditions

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_0}{\partial\rho}\right)+\frac{\partial^2\psi_0}{\partial\tilde{z}^2}=0\qquad\tilde{z}<0,\tag{40}$$

$$\psi_0 = 0 \qquad \tilde{z} = 0,\ \rho > 1, \tag{41}$$

$$\frac{\partial \psi_0}{\partial \tilde{z}} = a(t)[f_0(a\rho) - h(t)] \qquad \tilde{z} = 0,\ 0 < \rho < 1, \tag{42}$$

$$\frac{\partial \psi_0}{\partial \rho} = 0 \qquad \tilde{z} = 0,\ \rho = 1 - 0, \tag{43}$$

$$\psi_0 \to 0 \qquad (\tilde{z}^2 + \rho^2 \to \infty), \tag{44}$$

where $\psi_0(\rho, 0, t) = \Phi_0(\rho, t)$ for $\rho < 1$, the vertical coordinate $\tilde{z}$ is formal, its connection with the original variable z is complicated and is not discussed here. The axisymmetric solution of the mixed boundary-value problem (40)–(44) is detailed in Appendix A, it reads

$$\Phi_0(\rho, t) = \psi_0(\rho, 0, t) = \int_1^{1/\rho} \frac{\chi_a(\rho\nu, t)\,\mathrm{d}\nu}{\sqrt{\nu^2 - 1}} \qquad (0 < \rho < 1). \tag{45}$$

with

$$\chi_a(\mu, t) = \frac{2\mu a(t)}{\pi} \int_0^{\pi/2} \sin\beta[f_0(a\mu\sin\beta) - h(t)]\,\mathrm{d}\beta. \tag{46}$$

The Neumann boundary condition (43) along the unit circle gives

$$\begin{aligned}\frac{\partial \psi_0}{\partial \rho}(1-0, 0, t) &= \lim_{\rho\to 1-0} \frac{\partial \psi_0}{\partial \rho}(\rho, 0, t)\\ &= \lim_{\rho\to 1-0}\left[-\frac{\chi_a(1, t)}{\rho\sqrt{1-\rho^2}} + \int_1^{1/\rho} \frac{\nu(\partial\chi_a/\partial\mu)(\rho\nu, t)\,\mathrm{d}\nu}{\sqrt{\nu^2-1}}\right] = 0. \end{aligned}\tag{47}$$

This condition is satisfied if and only if $\chi_a(1, t) = 0$, which together with (46) leads to the equation

$$\int_0^{\pi/2} \sin\beta f_0\big(a(t)\sin\beta\big)\mathrm{d}\beta = h(t). \tag{48}$$

It can be shown that not only $(\partial\psi_0/\partial\rho)(\rho, 0, t)$, but also the derivative $(\partial\psi_0/\partial\tilde{z})(\rho, 0, t)$ are continuous as long as $\chi_a(1, t) = 0$. It is worth noting that equations (46)–(45) give the solution of the most general axisymmetric Wagner problem for an arbitrary axisymmetric shape of the entering body and for an arbitrary time variation of its penetration depth.

Equation (48) is equivalent to the classical Wagner condition, but is much simpler. This equation directly connects the body shape, the penetration depth and the radius of the contact region $a(t)$. It does not involve hydrodynamics and does not depend on the history of the process. The remarkable feature of the present formulation is that this single non–linear algebraic equation is simple to use for calculating the time dependence of the contact region size. To illustrate the next developments, we consider the case of the paraboloid, $f_0(r) = r^2/(2R)$, for which equation (48) gives $a(t) = \sqrt{3Rh(t)}$ which coincides with the previous results (see, e.g. Korobkin & Pukhnachov 1988). More generally, if $f_0(r) = A(r/r_c)^\gamma$, where A, r_c and γ are given constants and $\gamma \geqslant 1$, then

$$a(t) = r_c\left[\frac{h(t)}{Az_\gamma}\right]^{1/\gamma}, \qquad z_\gamma = \int_0^{\pi/2} \sin^{\gamma+1}\beta\,\mathrm{d}\beta. \tag{49}$$

In order to evaluate the function $Q_0(\rho, \alpha, t)$ in (36), which is dependent on the zeroth-order solution, we must find first the function $S_0(\rho, t)$ given by (31), where

$\Phi_0(\rho, t)$ is defined by (45). It is not difficult to obtain this function in the general case, but the corresponding form of the first-order solution will be complicated. Below, a particular case

$$f_0(r) = \sum_{n=1}^{\infty} \kappa_n r^n, \tag{50}$$

where the coefficients κ_n are given constants, is considered. With proper truncation, the polynomial expression (50) is sufficient to represent axisymmetric shapes of practical interest. It should be noted that the technique used below can also be applied to the case, where r^n in (50) is replaced by r^{γ_n} with $\gamma_1 \geqslant 1$ and $\gamma_{n+1} > \gamma_n$.

By substituting expression (50) into equations (45), (46) and (48), we obtain the relation between the time variations of $a(t)$ and $h(t)$

$$h(t) = \sum_{k=1}^{\infty} \kappa_k a^k z_k \tag{51}$$

and the displacement potential in the contact region $(0 < \rho < 1)$

$$\Phi_0(\rho, t) = -\frac{2}{\pi} \sum_{k=1}^{\infty} k\kappa_k z_k a^{k+1}(t) D_k(\rho), \qquad D_k(\rho) = \int_\rho^1 \nu^{k-1}\sqrt{\nu^2 - \rho^2}\, d\nu. \tag{52}$$

The planar Laplacian of Φ_0 is detailed in Appendix B.

5. First-order solution

After the zeroth-order functions $S_0(\rho, t)$ and $a(t)$ have been determined, the boundary-value problem (33)–(35) for the first-order unknown functions $S_1(\rho, \alpha, t)$, $\Phi_1(\rho, \alpha, t)$ and $H_0(\omega, t)$ can be solved. The analytic function $g(\omega, t)$ provides the conformal mapping of the unit circle $|\omega| < 1$ onto the contact region $\mathrm{D}(t)$. Therefore, $H_0(\omega, t)$ can be sought in the form

$$H_0(\omega, t) = \sum_{n=1}^{\infty} a_n(t)\omega^n \qquad (|\omega| < 1), \tag{53}$$

with unknown complex coefficients $a_n(t)$. The condition on the imaginary part of the mapping derivative: $\mathrm{Im}[(\partial g/\partial\omega)(0, t)] = 0$, gives $\mathrm{Im}(a_1) = 0$. By using the representations $\omega = \rho \mathrm{e}^{\mathrm{i}\alpha}$ and $\omega_0 = \rho_0 \mathrm{e}^{\mathrm{i}\alpha_0}$, we obtain

$$\frac{1}{\rho}\mathrm{Re}(\bar{\omega} H_0(\omega, t)) = \mathrm{Re}\sum_{n=0}^{\infty} \rho^{n+1} a_{n+1}(t)\mathrm{e}^{\mathrm{i}n\alpha}, \tag{54}$$

and

$$\mathrm{Re}(T_0(\omega, \omega_0, t)) = \mathrm{Re}\sum_{n=0}^{\infty}\left[a_{n+1}(t)\sum_{m=0}^{n} \rho^{n-m}\rho_0^m \mathrm{e}^{\mathrm{i}(n\alpha + m(\alpha_0 - \alpha))}\right]. \tag{55}$$

Details of the development are given in Appendix C.

The Fourier series of the shape function $F(r, \theta)$

$$F(r, \theta) = \mathrm{Re}\sum_{n=0}^{\infty} A_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{56}$$

where the complex coefficients $A_n(r)$, $n \geqslant 0$, are assumed to be given and $\mathrm{Im}A_0(r) \equiv 0$, is used below. Substituting equalities (54) and (55) into equation (36), we obtain the following Fourier series of the function $Q_0(\rho, \alpha, t)$

$$Q_0(\rho, \alpha, t) = \mathrm{Re} \sum_{n=0}^{\infty} Q_{0n}(\rho, t) \mathrm{e}^{\mathrm{i}n\alpha} \tag{57}$$

with

$$Q_{0n}(\rho, t) = a_{n+1} q_{0n}(\rho, t) - a A_n(a\rho) \qquad \text{for } n \geqslant 0 . \tag{58}$$

For the zeroth mode, $n = 0$, we find

$$q_{00}(\rho, t) = a(t)[h(t) - f_0(a\rho) - a\rho f_0'(a\rho)], \tag{59}$$

and for higher modes, $n \geqslant 1$,

$$q_{0n}(\rho, t) = M_n(\rho, t) - a^2(t) f_0'(a\rho)\rho^{n+1}, \tag{60}$$

where $M_n(\rho, t)$ is developed in Appendix C and has the form

$$M_n(\rho, t) = \frac{2}{\pi}\rho^n \left[ah\hat{z}_0^n - \sum_{k=1}^{\infty}(k+1)\kappa_k z_k \hat{z}_k^n a^{k+1} \rho^k \right], \qquad \hat{z}_N^n = \sum_{m=0}^{n} z_{2m+N-1}. \tag{61}$$

In order to evaluate the first-order displacement potential $\Phi_1(\rho, \alpha, t)$, we adopt the same procedure as for the zeroth-order solution in §4. The restriction of ψ_1 on the plane $z = 0$ is denoted Φ_1 and equation (57) shows that the unknown function $\psi_1(\rho, \alpha, \tilde{z}, t)$ has the form

$$\psi_1(\rho, \alpha, \tilde{z}, t) = \mathrm{Re} \sum_{n=0}^{\infty} \psi_{1n}(\rho, \tilde{z}, t) \mathrm{e}^{\mathrm{i}n\alpha}, \tag{62}$$

where the new unknown complex functions $\psi_{1n}(\rho, \tilde{z}, t)$ satisfy the following equations

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_{1n}}{\partial\rho}\right) + \frac{\partial^2\psi_{1n}}{\partial\tilde{z}^2} - \frac{n^2}{\rho^2}\psi_{1n} = 0, \qquad \tilde{z} < 0, \tag{63}$$

$$\psi_{1n} = 0, \qquad \tilde{z} = 0, \rho > 1, \tag{64}$$

$$\frac{\partial\psi_{1n}}{\partial\tilde{z}} = -Q_{0n}(\rho, t), \qquad \tilde{z} = 0, 0 < \rho < 1, \tag{65}$$

$$\frac{\partial\psi_{1n}}{\partial\rho} = 0, \qquad \tilde{z} = 0, \rho = 1 - 0, \tag{66}$$

$$\psi_{1n} \to 0 \qquad (\tilde{z}^2 + \rho^2 \to \infty). \tag{67}$$

The general solution of the mixed boundary-value problem (63)–(67) is detailed in Appendix A, it reads

$$\psi_{1n}(\rho, 0, t) = \int_1^{1/\rho} \frac{\chi_n(\rho\nu, t)\,\mathrm{d}\nu}{\nu^n\sqrt{\nu^2 - 1}} \qquad (0 < \rho < 1), \tag{68}$$

with

$$\chi_n(\mu, t) = -\frac{2\mu}{\pi}\int_0^{\pi/2}(\sin\beta)^{n+1} Q_{0n}(\mu\sin\beta, t)\,\mathrm{d}\beta. \tag{69}$$

It can be shown (see §4 for details) that the Neumann boundary condition (66) along the unit circle $\rho = 1$ is satisfied if and only if $\chi_n(1, t) = 0$. This equation can be represented using (69) as

$$\int_0^{\pi/2}(\sin\beta)^{n+1} Q_{0n}(\sin\beta, t)\,\mathrm{d}\beta = 0 \qquad (n \geqslant 0). \tag{70}$$

As described in Appendix D, equation (70) provides the coefficients a_n of $H_0(\omega, t)$ as

$$a_{n+1}(t) = -\frac{\dot{a}}{a\dot{h}} \int_0^{\pi/2} (\sin\beta)^{n+1} A_n(a \sin\beta) d\beta \qquad (n \geqslant 0). \tag{71}$$

In an important particular case, where the coefficients in series (56) are presented as

$$A_n(r) = \sum_{k=1}^{\infty} A_{nk} r^k \qquad (n \geqslant 0), \tag{72}$$

formula (71) gives

$$a_{n+1}(t) = -\frac{\displaystyle\sum_{k=1}^{\infty} A_{nk} z_{n+k} a^k(t)}{\displaystyle\sum_{k=1}^{\infty} k\kappa_k z_k a^k(t)}. \tag{73}$$

The coefficients A_{nk} in (72) and the functions $a_{n+1}(t)$ are complex, but the function $a(t)$ is real. From decomposition (72) the stability of the axisymmetric solution can be performed; this is described in Appendix E.

Once the complex functions $a_{n+1}(t)$ have been determined, the position of the contact line $\Gamma(t)$ is described in the first-order approximation by the equation

$$x + \mathrm{i}y = g(\mathrm{e}^{\mathrm{i}\alpha}, t) = a(t)[\mathrm{e}^{\mathrm{i}\alpha} + \epsilon H_0(\mathrm{e}^{\mathrm{i}\alpha}, t) + O(\epsilon^2)], \tag{74}$$

where H_0 follows from (53) and the angle α, $0 \leqslant \alpha < 2\pi$, is considered as the parameter.

The first-order displacement potential is reconstructed from equation (62) where $\psi_{1n}(\rho, 0, t)$ is given by equations (69) and (68). By using expansion (72), we obtain

$$\psi_{1n}(\rho, 0, t) = -\frac{2}{\pi}\rho^n a(t)\left[a_{n+1}(t)\sum_{k=1}^{\infty} k(k+1)\kappa_k z_k a^k(t) D_k(\rho) - \sum_{k=1}^{\infty}(k-n) A_{nk} z_{n+k} a^k(t) D_{k-n}(\rho)\right], \tag{75}$$

where the functions $D_n(\rho)$ are defined by (52). Formulae (71)–(75) provide the first-order solution of the original Wagner problem of impact.

6. Hydrodynamic force

As will be shown later in the text, the first-order solution does not contribute to the force. The leading-order contribution is of second order. Another correction could also be taken into account by integrating the uniformly valid distribution of pressure which usually originates from the local matching between the main flow solution and a local solution valid at the spray root. This will be done in future works.

The vertical component of the hydrodynamic force $F(t)$ on the entering body is given by formula (20), where the double integral is denoted $N(t, \epsilon)$ below. We shall determine the asymptotics of the function $N(t, \epsilon)$ as $\epsilon \to 0$.

By using expansions (22), we obtain

$$\begin{aligned} N(t, \epsilon) &= \iint_{\rho<1} \Phi(\rho, \alpha, t) |(\partial g/\partial\omega)(\omega, t)|^2 \rho \,\mathrm{d}\rho \,\mathrm{d}\alpha \\ &= a^2[N_0(t) + \epsilon N_1(t) + \epsilon^2 N_2(t) + O(\epsilon^3)], \end{aligned} \tag{76}$$

where

$$N_0(t) = \iint_{\rho<1} \Phi_0(\rho, t)\rho \mathrm{d}\rho \mathrm{d}\alpha, \tag{77}$$

$$N_1(t) = \iint_{\rho<1} [\Phi_1(\rho, \alpha, t) + 2\Phi_0(\rho, t)\mathrm{Re}(H_0')]\rho \mathrm{d}\rho \mathrm{d}\alpha, \tag{78}$$

$$N_2(t) = N_{20}(t) + N_{21}(t) + N_{22}(t), \tag{79}$$

$$N_{20}(t) = \iint_{\rho<1} \Phi_0(\rho, t)[2\mathrm{Re}(H_1') + |H_0'|^2]\rho \mathrm{d}\rho \mathrm{d}\alpha, \tag{80}$$

$$N_{21}(t) = 2\iint_{\rho<1} \Phi_1(\rho, \alpha, t)\mathrm{Re}(H_0')\rho \mathrm{d}\rho \mathrm{d}\alpha, \tag{81}$$

$$N_{22}(t) = \iint_{\rho<1} \Phi_2(\rho, \alpha, t)\rho \mathrm{d}\rho \mathrm{d}\alpha. \tag{82}$$

Notice that the first-order contribution, $N_1(t)$, is trivial. Indeed, equations (62), (75) and (53) yield that $N_1(t)$ is proportional to $a_1(t)$, which is zero if $A_0(r) \equiv 0$ as it follows from formula (71). It is convenient to redefine decomposition (1) in such a way that

$$f_0(r) = \langle f(x, y)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, r\sin\theta))\,\mathrm{d}\theta, \tag{83}$$

where $\langle . \rangle$ stands for the average with respect to the angular coordinate. This definition of the axisymmetric part $f_0(r)$ in decomposition (1) is used below. With this definition we have $A_0(r) \equiv 0$ in the Fourier series (56) and, therefore, $N_1(t) \equiv 0$.

The leading-order term in expansion (76) is computed with the help of equation (52)

$$N_0(t) = -\frac{4}{3}\sum_{n=1}^{\infty} \frac{n}{n+3}\kappa_n z_n a^{n+1}(t). \tag{84}$$

Equation (84) leads to the well-known formula for the hydrodynamic force on an axisymmetric body entering the liquid vertically

$$F_a(t) = \frac{\mathrm{d}}{\mathrm{d}t}[m_a(t)\dot{h}], \tag{85}$$

where $m_a(t) = 4\rho_0 a^3(t)/3$ is the added mass of the circular disk of radius $a(t)$. We obtain

$$F(t) = -\rho_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^2 N_0(t)] + O(\epsilon^2) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{4}{3}\rho_0\sum_{n=1}^{\infty} n\kappa_n z_n a^{n+2}(t)\dot{a}\right] + O(\epsilon^2). \tag{86}$$

Differentiating (51) in time and substituting the result into the latter expression, we arrive at

$$F(t) = F_a(t) - \rho_0\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^2 N_2(t)]\epsilon^2 + O(\epsilon^3). \tag{87}$$

In order to evaluate te function $N_2(t)$ and to obtain the second-order contribution to the hydrodynamic force as $\epsilon \to 0$, the second-order solution of the impact problem is required. However, two integrals from (79) can be calculated using the zeroth- and first-order solutions

$$N_{21}(t) = -2a\sum_{k=1}^{\infty} k(k+1)\kappa_k z_k a^k \left(\sum_{n=2}^{\infty} \frac{z_{2n}|a_n|^2}{2n+k+1}\right) - 2a\mathrm{Re}\sum_{k=1}^{\infty} z_{2k+2}\bar{a}_{k+1}\left(\sum_{n=1}^{\infty}\frac{n-k}{n+k+3}A_{kn}z_{n+k}a^n\right), \qquad (88)$$

$$\iint_{\rho<1} \Phi_0(\rho,t)|H_0'|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha = -2a\sum_{k=1}^{\infty} k\kappa_k z_k a^k \left(\sum_{n=2}^{\infty}\frac{n z_{2n}|a_n|^2}{2n+k+1}\right). \qquad (89)$$

The second-order solution satisfies equations (37)–(39), where $j=2$ and

$$K_2(\rho,\alpha,t) = -\frac{1}{2\pi a}\iint_{|\omega_0|<1}\frac{S_1(\xi_0,\eta_0,t)\mathrm{Re}(T_0)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega-\omega_0|} - \frac{1}{2\pi a}\iint_{|\omega_0|<1}\left[\tfrac{1}{2}|T_0|^2 - \tfrac{3}{2}\mathrm{Re}^2(T_0)\right]\frac{S_0(\xi_0,\eta_0,t)\,\mathrm{d}\xi_0\,\mathrm{d}\eta_0}{|\omega-\omega_0|} - \tilde{Y}_2(\xi,\eta,t). \qquad (90)$$

The function $\tilde{Y}_2(\xi,\eta,t)$ is given by equations (28) and (29) as

$$\tilde{Y}_2(\xi,\eta,t) = \frac{a}{2\rho}f_0'(a\rho)\left[\frac{1}{\rho^2}\mathrm{Re}^2(\bar{\omega}H_0) - |H_0|^2\right] - \frac{a^2}{2\rho^2}f_0''(a\rho)\mathrm{Re}^2(\bar{\omega}H_0) - \frac{a}{\rho}F_{,r}(a\rho,\alpha)\mathrm{Re}(\bar{\omega}H_0) - F_{,\theta}(a\rho,\alpha)\,\mathrm{Im}(H_0/\omega). \qquad (91)$$

It can be seen that the forcing term in equation (37) is complicated. We do not expect that higher-order solutions can be developed in analytical forms for an arbitrary shape of the body. However, generally speaking, at any order with respect to ϵ, the solution is given by equations (71) and (75), where the coefficients $A_n(r)$ are to be replaced by the corresponding coefficients in the Fourier series of $K_j(\rho,\alpha,t)$ (see equation (56)). Fortunately, in order to calculate the second-order contribution to the vertical component of the hydrodynamic force, we need not obtain the second-order solution in full detail.

The second-order solution is sought in the form

$$H_1(\omega,t) = \sum_{n=1}^{\infty} b_n(t)\omega^n \qquad (|\omega|<1), \qquad (92)$$

$$\Phi_2(\rho,\alpha,t) = \mathrm{Re}\sum_{n=0}^{\infty}\Phi_{2n}(\rho,t)\mathrm{e}^{in\alpha}, \quad S_2(\rho,\alpha,t) = \mathrm{Re}\sum_{n=0}^{\infty} S_{2n}(\rho,t)\mathrm{e}^{in\alpha}. \qquad (93)$$

Substituting (92) and (93) into equations (80), (82) and using (77), we find

$$\iint_{\rho<1}\Phi_0(\rho,t)\mathrm{Re}(H_1')\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha = 2\pi b_1(t)N_0(t), \qquad (94)$$

$$\iint_{\rho<1}\Phi_2(\rho,\alpha,t)\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha = 2\pi\int_0^1\Phi_{20}(\rho,t)\rho\,\mathrm{d}\rho. \qquad (95)$$

Therefore, in order to evaluate the functions $N_{20}(t)$ and $N_{22}(t)$, it is only required to determine $b_1(t)$ and $\Phi_{20}(\rho,t)$ but not the total second-order solution. These functions are given by formulae (71) and (80), where $A_0(r)$ has to be replaced by $\langle K_2(r,\alpha,t)\rangle$.

Combining equations (76)–(95), we arrive at the second-order asymptotics of the function $N(t,\epsilon)$ and, using (20), at the second-order hydrodynamic force on the body entering a liquid vertically. Calculations are shown in §§ 7 and 8 for particular non-axisymmetric shapes of entering bodies.

The horizontal components, $F_x(t)$ and $F_y(t)$, of the hydrodynamic force acting on the entering body is given with the Wagner theory as

$$\left.\begin{aligned} F_x(t) &= \rho_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathrm{D}(t)} f'_x(x,y)\phi(x,y,0,t)\,\mathrm{d}x\,\mathrm{d}y, \\ F_y(t) &= \rho_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathrm{D}(t)} f'_y(x,y)\phi(x,y,0,t)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}\right\} \tag{96}$$

By using the conformal mapping of the wetted region $\mathrm{D}(t)$ onto the unit circular disk (see § 2), equations (96) are turned into

$$(F_x(t), F_y(t)) = \rho_0 \frac{\mathrm{d}^2}{\mathrm{d}t^2} \iint_{\rho<1} \Phi(\rho,\alpha,t)|(\partial g/\partial\omega)(\omega,t)|^2 \nabla f(g_1(\xi,\eta,t), g_2(\xi,\eta,t))\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha, \tag{97}$$

where $\nabla f = (f_{,x}, f_{,y})$. The double integral in (97) is denoted by $\boldsymbol{P}(t,\epsilon)$. The asymptotic expansion of the vector-function $\boldsymbol{P}(t,\epsilon)$ as $\epsilon \to 0$ has the form

$$\boldsymbol{P}(t,\epsilon) = a^2(t)[\boldsymbol{P}_0(t) + \epsilon \boldsymbol{P}_1(t) + \epsilon^2 \boldsymbol{P}_2(t) + O(\epsilon^3)], \tag{98}$$

where

$$\boldsymbol{P}_0(t) = \iint_{\rho<1} \Phi_0(\rho,t) f'_0(a\rho)\boldsymbol{e}_1(\alpha)\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha, \tag{99}$$

$$\boldsymbol{P}_1(t) = \boldsymbol{P}_{11}(t) + \boldsymbol{P}_{12}(t), \tag{100}$$

$$\boldsymbol{P}_{11}(t) = \iint_{\rho<1} [\Phi_1(\rho,\alpha,t) + 2\Phi_0(\rho,t)\mathrm{Re}(H'_0)] f'_0(a\rho)\boldsymbol{e}_1(\alpha)\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha, \tag{101}$$

$$\boldsymbol{P}_{12}(t) = \iint_{\rho<1} \Phi_0(\rho,t)[L_1(\rho,\alpha,t)\boldsymbol{e}_1(\alpha) + L_2(\rho,\alpha,t)\boldsymbol{e}_2(\alpha)]\rho\,\mathrm{d}\rho\,\mathrm{d}\alpha, \tag{102}$$

$$\begin{aligned} L_1(\rho,\alpha,t) &= \frac{a}{\rho} f''_0(a\rho)\mathrm{Re}(\bar{\omega}H_0) + F_{,r}(a\rho,\alpha), \\ L_2(\rho,\alpha,t) &= f'_0(a\rho)\,\mathrm{Im}(H_0/\omega) + \frac{1}{a\rho}F_{,\theta}(a\rho,\alpha), \end{aligned} \tag{103}$$

$$\boldsymbol{e}_1(\alpha) = (\cos\alpha, \sin\alpha), \qquad \boldsymbol{e}_2(\alpha) = (-\sin\alpha, \cos\alpha). \tag{104}$$

It is clear that $\boldsymbol{P}_0(t) \equiv 0$. By using equations (52), (73), (75) and (100)–(104), we derive the formula

$$\boldsymbol{P}_1(t) = \mathrm{Re}\{\pi(1,i)\tilde{P}(t)\}, \tag{105}$$

$$\tilde{P}(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n}{n+3}\kappa_n z_n a^n \left(\sum_{k=2}^{\infty} \frac{k-1}{k+n+2} A_{1k} z_{k+1} a^k\right), \tag{106}$$

which presents the leading-order horizontal force as $\epsilon \to 0$ with the help of the coefficients in expansions (50) and (72). The function $a(t)$ in (106) is the solution of equation (51), where the penetration depth $h(t)$ is assumed given. Equation (106) shows that the first-order horizontal force is different from zero if and only if

$$\int_0^{2\pi} F(r,\theta)e^{i\theta}d\theta \neq 0, \quad A_1(r) \neq A_{11}r. \tag{107}$$

As an example of the horizontal force calculations, consider the body, the shape of which is described by the equation

$$z = r\tan\gamma + \epsilon\frac{r^2}{2R}\cos\theta, \tag{108}$$

where γ is the deadrise angle at the lowest point of the body, $r = 0$, and R and ϵ are given quantities. For shape (108), equations (87) and (106) provide the leading-order force components as

$$F_z(t) = \frac{64}{3\pi^3}\rho_0\tan^{-3}\gamma\frac{d^2}{dt^2}[h^4(t)](1 + O(\epsilon^2)), \tag{109}$$

$$F_x(t) = \frac{24\epsilon}{5\pi^3 R}\rho_0\tan^{-4}\gamma\frac{d^2}{dt^2}[h^5(t)](1 + O(\epsilon)), \tag{110}$$

$$F_y(t) = 0. \tag{111}$$

Equation of the horizontal body motion, $M\ddot{x} = F_x(t)$, where M is the body mass, can be integrated twice

$$x(t) = \frac{24\epsilon}{5\pi^3 RM}\rho_0\tan^{-4}\gamma h^5(t)(1 + O(\epsilon)). \tag{112}$$

For a body with parameters $\tan\gamma = 0.1$, $R = 5$ m, $\epsilon = 0.1$ and $M = 10$ kg, equation (112) predicts the horizontal displacement of the body 1 cm for the penetration depth h of 10 cm. The horizontal displacement of an asymmetric body entering liquid vertically can be measured in experiments. We are unaware of any such experiments.

7. Elliptic paraboloid impact

The shape function $f(x, y)$ of an elliptic paraboloid is given as

$$f(x,y) = \frac{x^2}{2r_x} + \frac{y^2}{2r_y}, \tag{113}$$

where r_x and r_y, $r_y \geqslant r_x$, are the radii of the body curvature at the impact point $x = 0$, $y = 0$. Decomposition (1) and equation (83) provide

$$f_0(r) = \frac{r^2}{2R}, \qquad F(r,\theta) = \frac{r^2}{2R}\cos 2\theta, \tag{114}$$

where

$$R = \frac{2r_x r_y}{r_x + r_y}, \qquad \epsilon = \frac{r_y - r_x}{r_y + r_x}. \tag{115}$$

We consider here the case where $\epsilon \ll 1$. In expansion (50), $\kappa_2 = 1/(2R)$ and $\kappa_n = 0$ for $n \neq 2$. Equations (48) and (52) give after some algebra

$$a(t) = \sqrt{3Rh(t)}, \qquad \Phi_0(\rho, t) = -\frac{4ha}{3\pi}(1 - \rho^2)^{3/2}. \tag{116a, b}$$

The first formula for the radius of the contact line in the paraboloid entry problem is well-known (see Korobkin & Pukhnachov 1988). However, (116b) for the zeroth-order displacement potential is not easy to recognize. In order to demonstrate that this formula provided the well-known velocity potential in the axisymmetric contact region, we have to rewrite it in terms of the original variables

$$\tilde{\varphi}_0(r,t) = -\frac{4}{9\pi R}(a^2(t)-r^2)^{3/2} \qquad (0<r<a(t)) \tag{117}$$

and calculate the zeroth-order velocity potential

$$\phi_0(r,t) = \frac{\partial \tilde{\varphi}_0}{\partial t}(r,t) = -\frac{2\dot{h}}{\pi}\sqrt{a^2(t)-r^2} \qquad (0<r<a(t)). \tag{118}$$

The latter formula is in agreement with the results to Schmieden (1953).

In the present case, only the real part of the coefficient A_2 is different from zero in expansion (56), $A_2 = r^2/(2R)$. Correspondingly, in equation (72), $A_{22} = 1/(2R)$ and $A_{2k} = 0$ for $k \neq 2$. Equation (73) gives that only $a_3(t)$ is different from zero in the first-order conformal mapping (53)

$$a_3(t) = -\tfrac{2}{5}. \tag{119}$$

Therefore, the contact line at the first-order approximation is described by the equation

$$r_{\Gamma(t)}(\theta,t) = \sqrt{3Rh(t)}\left[1 - \tfrac{2}{5}\epsilon\cos(2\theta) + O(\epsilon^2)\right], \tag{120}$$

which coincides with the corresponding asymptotics obtained from the exact solution by Scolan & Korobkin (2001). Details can be found in Appendix F.

Equation (B 1) yields

$$S_0(\rho,t) = \frac{4a^3}{3\pi R}\left[3\sqrt{1-\rho^2} - \frac{1}{\sqrt{1-\rho^2}}\right], \tag{121}$$

and formula (61) takes the form

$$M_2(\rho,t) = \frac{a^3(t)\rho^2}{R}\left(\tfrac{5}{8} - \tfrac{19}{16}\rho^2\right). \tag{122}$$

Equations (122) and (90), (91) provide

$$\langle \tilde{Y}_2(\xi,\eta,t)\rangle = -\frac{2a^2\rho^6}{25R}, \tag{123}$$

$$\langle K_2(\rho,\alpha,t)\rangle = \frac{a^2}{3200R}[1225\rho^6 - 750\rho^4 - 168\rho^2 - 48], \tag{124}$$

and (71) gives $b_1(t) \equiv 0$. By algebra

$$N_0(t) = -\frac{8}{45}\frac{a^3(t)}{R}, \qquad N_2(t) = -\frac{8}{75}\frac{a^3(t)}{R}, \tag{125}$$

and the second-order hydrodynamic force on the entering elliptic paraboloid is obtained as

$$F(t) = \frac{8\rho_0}{45R}\left(1 + \tfrac{3}{5}\epsilon^2 + O(\epsilon^3)\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^5(t)]. \tag{126}$$

The asymptotic formula (126) is in agreement with the exact formula,

$$F(t) = \frac{\rho_0 G(\epsilon)}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^5(t)], \tag{127}$$

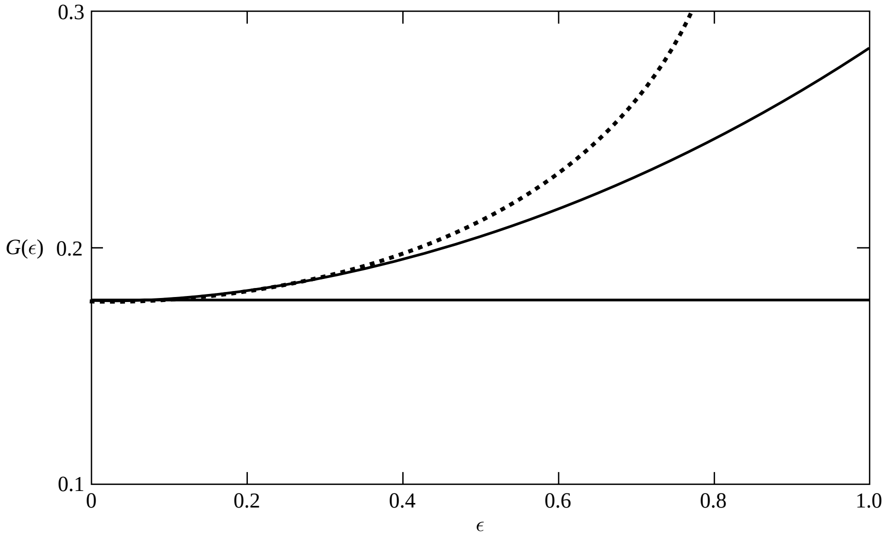

FIGURE 3. Variation of the force coefficient $G(\epsilon)$ with the small parameter ϵ. Dotted line, exact solution derived in Scolan & Korobkin (2001); thin solid line, zeroth-order approximation; thick solid line, second-order approximation (see equation (126)).

derived by Scolan & Korobkin (2001) for the elliptic paraboloid entry problem, where the function $G(\epsilon)$ was obtained in analytical form for $0 \leqslant \epsilon < 1$. Asymptotic analysis of the function $G(\epsilon)$ shows that the coefficient $8/45[1 + 3/5\epsilon^2 + O(\epsilon^3)]$ in (126) is exactly the second-order asymptotics of the function $G(\epsilon)$ from (127) as $\epsilon \to 0$, which justifies the present asymptotic analysis against the exact solution.

The function $G(\epsilon)$ (dotted line) together with its zeroth-order (thin solid line) and second order (thick solid line) approximations is shown in figure 3. It can be seen that the second-order approximation (126) of the hydrodynamic force can be used with less than 5 % relative error up to $\epsilon = 0.56$.

It should be noted, on the other hand, that the asymptotic solution obtained in this study justifies the exact solution, which was derived on the assumption that the contact line for an elliptic paraboloid entering liquid is elliptic. In the present study, this assumption is not employed.

8. Vertical entry of a slightly inclined cone

Vertical entry of an inclined cone with a deadrise angle of 15° was studied experimentally by Shorygin (1973). The cone was of small height H. Experimental results and discussion on them are presented for angles α of the cone inclination from 5° to 15°. The methodology for both experiments and measurements is not given. Results are presented, in particular, for maximum value of the axial component of the resistance force. It was observed in the experiments that these maximum values occur at the time instant, when the contact line reaches the edge of the cone base. This implies that the Wagner approach can be used to estimate the axial force. It was determined that the inclination of the cone at an angle of 5° reduces the axial force in the range of [19.67%, 24.6%] compared to the value of the axial force at zero inclination angle.

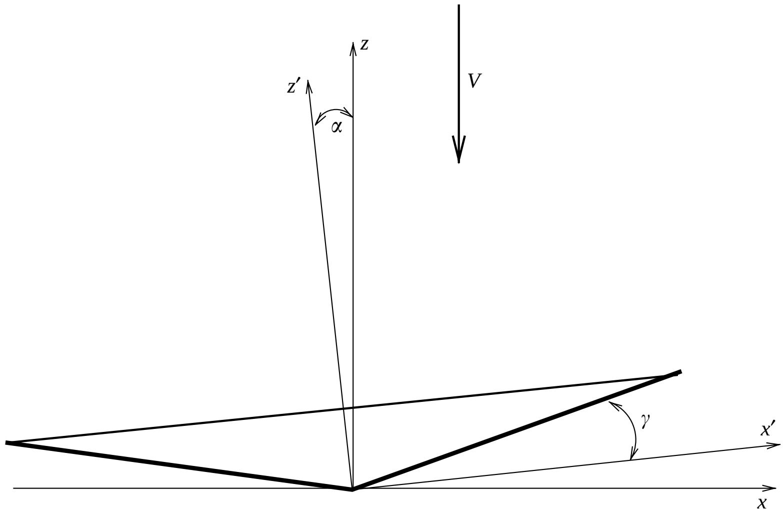

FIGURE 4. The inclined cone enters the liquid, $z < 0$, vertically with a constant velocity V. The inclination angle α is measured between the vertical axis z and the cone axis z'. The deadrise angle γ is measured from the x'-axis. In the present asymptotic study $\alpha \ll \gamma$.

We shall apply the asymptotic theory developed in the previous sections to the problem of inclined cone entry and estimate reduction of the axial resistance force for the conditions of the experiments by Shorygin (1973).

A cone is described in the coordinate system (O, x', y', z') by the equation (see figure 4)

$$z' = T\sqrt{x'^2 + y'^2} \quad \text{with} \quad T = \tan\gamma. \tag{128}$$

The cone is inclined in the plane Oxz at small angle α as shown in figure 4.

The hydrodynamic force, which acts on the entering cone, has two components: the vertical component $F_z(t, \alpha, \gamma)$ and the horizontal component $F_x(t, \alpha, \gamma)$. The axial force on the cone $F_{z'}(t, \alpha, \gamma)$ is the projection of the total force vector on the cone axis Oz' and is given as

$$F_{z'}(t, \alpha, \gamma) = F_z(t, \alpha, \gamma)\cos\alpha - F_x(t, \alpha, \gamma)\sin\alpha, \tag{129}$$

where $F_x(t, 0, \gamma) = 0$ and $F_{z'}(t, 0, \gamma) = F_z(t, 0, \gamma)$.

The equation of the inclined cone is given in the coordinate system (O, x, y, z) as $z = rTN(\alpha, \gamma, \theta)$, where $N(\alpha, \gamma, \theta)$ is the solution of the quadratic equation

$$N^2 = 1 + 2\frac{\sin(2\alpha)}{\sin(2\gamma)}N\cos\theta - \frac{\sin^2\alpha}{\sin^2\gamma}\cos^2\theta + N^2\frac{\sin^2\alpha}{\cos^2\gamma}, \tag{130}$$

which tends to unity as $\alpha \to 0$. For small inclination angle α, equation (130) exhibits the small parameter

$$\epsilon = \frac{\sin(2\alpha)}{\sin(2\gamma)}, \tag{131}$$

which implies that the inclined cone deviates slightly from the corresponding axisymmetric body only if $\alpha \ll \gamma$. Under experimental conditions, we roughly satisfy these assumptions since $\gamma = 15^\circ$ and $\alpha = 5^\circ$ yielding $\epsilon = 0.3473$. The coefficient T in

(128) is equal to 0.268 and is considered together with ϵ as a small parameter of the problem.

The asymptotic solution of equation (130) for $\epsilon \to 0$ has the form

$$z = rT\left[1 + \epsilon\cos\theta + \tfrac{1}{4}\epsilon^2\sin^2\gamma(3+\cos 2\theta) + O(\epsilon^3)\right]. \qquad (132)$$

Note that in the asymptotic analysis, we do not assume that the deadrise angle γ is small.

Comparing (132) with decomposition (1), we obtain

$$f_0(r) = rT, \qquad (133)$$

$$F(r,\theta) = rT\left[\cos\theta + \tfrac{1}{4}\epsilon\sin^2\gamma(3+\cos 2\theta) + O(\epsilon^3)\right]. \qquad (134)$$

Therefore, only the coefficient κ_1 differs from zero in (50), $\kappa_1 = T$. Equation (51) provides the radius of the contact line at the zeroth order as

$$a(t) = \frac{4h(t)}{\pi T}. \qquad (135)$$

The zeroth-order displacement potential follows from (52)

$$\Phi_0 = -\tfrac{1}{2}Ta^2 D_1(\rho). \qquad (136)$$

In the first-order analysis, only the real part of the coefficient A_1 is different from zero in expansion (56), $A_1 = rT$. Correspondingly, $A_{11} = T$ in (72) and $A_{nk} = 0$ for any other values of n and k. Equation (73) gives the coefficients of the conformal mapping

$$a_2(t) = -\frac{8}{3\pi}, \quad a_n(t) = 0 \quad (n \neq 2). \qquad (137)$$

The position of the contact line $\Gamma(t)$ is described in the first order by equation (74), which provides

$$r_{\Gamma(t)}(\theta,t) = \frac{4h(t)}{\pi T}\left(1 - \frac{8}{3\pi}\epsilon\cos\theta + O(\epsilon^2)\right). \qquad (138)$$

In order to evaluate the second-order vertical force component, we use equations (90) and (91) to find $\langle\tilde{Y}_2\rangle$ and $\langle K_2(\rho,\alpha,t)\rangle$. The coefficient b_1 is calculated with the help of the equation

$$b_1 = -\frac{\dot{a}}{a\dot{h}}\int_0^{\pi/2}\sin\beta\langle K_2(\sin\beta,\alpha,t)\rangle\,\mathrm{d}\beta, \qquad (139)$$

which gives

$$b_1 = -\tfrac{3}{4}\sin^2\gamma. \qquad (140)$$

This quantity is small for small deadrise angles and can be neglected.

The components (80) and (81) of the second-order vertical force are

$$N_0 = -\frac{\pi}{12}Ta^2, \quad N_{21} = -\frac{256}{405}\frac{Ta^2}{\pi}, \quad N_{20} = N_{21} + 2b_1N_0. \qquad (141)$$

The term N_{22} is obtained from equation (82) as

$$N_{22} = 2\pi\int_0^1 x\chi_0(x)\,\mathrm{d}x, \quad \chi_0(x) = \frac{2ax}{\pi}\int_0^{\pi/2}\sin\beta\langle K_2(x\sin\beta,\alpha,t)\rangle\,\mathrm{d}\beta, \qquad (142)$$

which leads to the formula

$$N_{22} = -\left(\frac{208}{405\pi} + \frac{3\pi}{16}\sin^2\gamma\right)Ta^2. \qquad (143)$$

Finally, the second-order vertical force on the inclined cone reads

$$F_z(t) = \rho_0 T\left(\frac{\pi}{12} + \epsilon^2\left[\frac{16}{9\pi} - \frac{\pi}{4}\sin^2\gamma\right]\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a^4(t)] + O(\epsilon^3). \tag{144}$$

The first-order horizontal component of the hydrodynamic force on the inclined cone is equal to zero, which follows from equation (106). Equation (129) for the axial force component provides in the case of constant entry velocity V_0, $h(t) = V_0 t$,

$$F_{z'}(t, \alpha, \gamma) = 12\rho_0 V_0^2 T\cos\alpha\left(\frac{\pi}{12} + \epsilon^2\left[\frac{16}{9\pi} - \frac{\pi}{4}\sin^2\gamma\right]\right)h^2(t). \tag{145}$$

We assume, following the idea by Shorygin (1973), that the maximum of the axial force $F_{z'}^{max}(\alpha, \gamma)$ occurs at the end of the impact stage, when the contact line (138) reaches the edge of the cone. Figure 4 shows that the contact line reaches the edge of the cone base at the time instant t_*, when the penetration depth $h(t_*)$ is equal to

$$h(t_*) = \frac{\pi}{4}H\frac{\cos\alpha + T\sin\alpha}{1 + 8\epsilon/(3\pi) + O(\epsilon^2)}. \tag{146}$$

By substituting this value into (145), we obtain the ratio

$$\delta = \frac{F_{z'}(t_*, 0, \gamma) - F_{z'}(t_*, \alpha, \gamma)}{F_{z'}(t_*, 0, \gamma)} \times 100\% \tag{147}$$

which was reported in the paper by Shorygin (1973). Calculations performed within the asymptotic theory developed in the present paper provide $\delta \approx 23.6\%$, which is well within the range [19.67%, 24.6%] obtained in experiments.

9. Water entry of a pyramid

It is questionable whether the present asymptotic analysis can be applied to a non-smooth body. In this section, the first-order solution is formally obtained for a pyramid, which is a non-smooth body. It is shown in the following development that the obtained contact line is smooth. A generalization of this theoretical result will deserve special attention in future works. However, numerical studies by Gazzola *et al.* (2005) seem to confirm this result.

We consider a pyramid with square cross-section

$$z = rw(\theta)\tan\gamma - h(t), \quad w(\theta) = \frac{\sqrt{8}}{\pi}\left(1 + \frac{2}{15}\sum_{m=1}^{\infty}\frac{15}{16m^2 - 1}(-1)^{m+1}\cos 4m\theta\right), \tag{148}$$

where γ is the deadrise angle, r, θ are the polar coordinates and $h(t)$ is the penetration depth. The small parameter is formally introduced as $\epsilon = 2/15$.

The pyramid entry problem is self-similar with the stretched variables r_1, z_1, θ introduced as

$$r_1 = rh^{-1}(t)\tan\gamma, \qquad z_1 = zh^{-1}(t)\tan\gamma. \tag{149}$$

In the new variables, the displacement potential (7) can be written in the form

$$\varphi(x, y, z, t) = h^2(t)\tan^{-1}\gamma\,\varphi_1(r_1, \theta, z_1), \tag{150}$$

where the new unknown function $\varphi_1(r_1, \theta, z_1)$ is independent of time t. In the stretched coordinates, the radius of the contact line is only dependent on the formal parameter

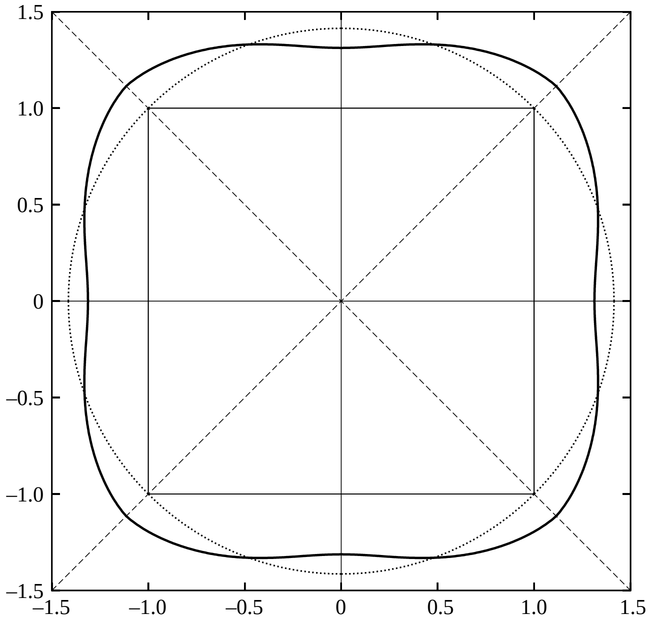

FIGURE 5. First-order approximation of the contact line (thick line) for a square pyramid entering the liquid vertically. Thin solid line, intersection of the pyramid with the undisturbed free surface; dotted line, zeroth-order approximation of the contact line.

ϵ and the angular coordinate θ. The first-order solution provides the contact line in the parametric form

$$x_1 + \mathrm{i}y_1 = \sqrt{2}\mathrm{e}^{\mathrm{i}\alpha} + \sum_{m=1}^{\infty} b_m \mathrm{e}^{\mathrm{i}[4m+1]\alpha}, \quad b_{m+1} = -b_m \frac{16m^2 - 1}{8(2m+3)(m+1)}, \quad b_1 = -\frac{\sqrt{2}}{12}, \tag{151}$$

where $0 \leqslant \alpha < 2\pi$. In figure 5, the contact line is shown with thick solid curve, the zeroth-order approximation provides the circle (dotted line), the square corresponds to the intersection line between the pyramid and the undisturbed free surface. The approximate contact line is smooth although the intersection line has corner points. Experiments with transparent pyramids are urgently required to confirm the results obtained.

The hydrodynamic vertical force on a pyramid with square cross-section is obtained by substituting (149) and (150) into equation (13) in the form

$$F(t) = \rho_0 N \tan^{-3}\gamma \frac{\mathrm{d}^2}{\mathrm{d}t^2}[h^4(t)], \tag{152}$$

where N is a universal constant which we suggest could be calculated using experimental data. If the pyramid is of rectangular cross-section, the constant N is dependent on the aspect ratio of the cross-sections. No attempt has been made to evaluate N neither analytically or numerically. We are unaware of any experimental study of the pyramid entry problem.

10. Conclusion

The main objective of the Wagner model is the determination of the contact line which bounds the wetted surface of the body. For almost axisymmetric bodies, we introduce a small parameter as a measure of the perturbation with regard to the nearest axisymmetric body. In this way, an elliptic paraboloid or a slightly inclined cone and pyramid can be considered as a perturbation of a sphere or a cone, respectively. This idea can be formalized mathematically by linearizing the classical Wagner problem on the basis of an axisymmetric solution via a perturbation technique.

In the present problem there is another small parameter which is the deadrise angle. It was not the purpose of this paper to evaluate higher-order contributions with respect to this small parameter. Only the leading-order solution has been obtained.

In order to perform the asymptotic analysis, the original problem is reduced to the integral equation (15) and the boundary-value problem for the Poisson equation (16). This formulation is expected to be helpful for developing a numerical algorithm to solve the impact problem for an arbitrary body shape.

The zeroth-order solution is well known. The first-order solution provides a first approximation of the contact line, but is does not provide any contribution to the forces. This is the reason for which the second-order problem must be partly solved in order to yield the next contribution to the force.

Both vertical and horizontal components of the hydrodynamic force on the entering body are obtained. It is also mentioned that the problem is studied within the classical Wagner theory without correction for nonlinear effects. These nonlinear corrections can be introduced either by using the uniformly valid distribution of pressure up to the spray root region, or by using the quadratic term of the pressure in Bernoulli equation.

The asymptotic results are compared with the exact solution in § 7 and with the experimental data in § 8. The comparisons show the efficiency of the approach developed here. It is important that the formulae obtained are in analytical forms and can be used for both optimization of the entering body shape and for study of the stability of the impact problem solutions.

Appendix A. Solutions of the mixed boundary-value problem

The general axisymmetric solution of the Laplace equation (40) in the lower half-space $\tilde{z} < 0$, which decays at infinity, has the form

$$\psi_0(\rho, \tilde{z}, t) = \int_0^\infty U_0(\lambda, t) \mathrm{e}^{\lambda \tilde{z}} J_0(\lambda \rho) \, \mathrm{d}\lambda, \tag{A 1}$$

where J_0 denotes the zeroth-order Bessel function of the first kind. Substituting (A 1) into the boundary conditions (41) and (42), we obtain the system of coupled integral equations

$$\left.\begin{aligned} &\int_0^\infty \lambda U_0(\lambda, t) J_0(\lambda \rho) \, \mathrm{d}\lambda = a(t)[f_0(a\rho) - h(t)] \qquad (0 < \rho < 1), \\ &\int_0^\infty U_0(\lambda, t) J_0(\lambda \rho) \, \mathrm{d}\lambda = 0 \qquad (\rho > 1). \end{aligned}\right\} \tag{A 2}$$

The function U_0 is calculated in Sneddon (1966)

$$U_0(\lambda, t) = \int_0^1 \chi_a(\mu, t)\sin(\mu\lambda)\,\mathrm{d}\mu, \tag{A 3}$$

$$\chi_a(\mu, t) = \frac{2a(t)}{\pi}\int_0^{\mu} \frac{\sigma\,[f_0(a\sigma) - h(t)]\,\mathrm{d}\sigma}{\sqrt{\mu^2 - \sigma^2}}, \tag{A 4}$$

which leads to (46) by using the change of variable $\sigma = \mu\sin\theta$. By combining equations (A 1) and (A 3), we arrive at

$$\psi_0(\rho, 0, t) = \int_1^{1/\rho} \frac{\chi_a(\rho\nu, t)\,\mathrm{d}\nu}{\sqrt{\nu^2 - 1}} \qquad (0 < \rho < 1). \tag{A 5}$$

The method of solution for the mixed boundary-value problem (63)–(67) is a generalization of the already treated zeroth-order boundary-value problem. Its general solution in the lower half-space $\tilde{z} < 0$, which vanishes as $\tilde{z} \to -\infty$, can be expressed as

$$\psi_{1n}(\rho, \tilde{z}, t) = \int_0^{\infty} U_n(\lambda, t)\mathrm{e}^{\lambda\tilde{z}} J_n(\lambda\rho)\,\mathrm{d}\lambda. \tag{A 6}$$

By substituting (A 6) into the boundary conditions (64) and (65), we obtain the system of coupled integral equations

$$\left.\begin{aligned} &\int_0^{\infty} \lambda U_n(\lambda, t) J_n(\lambda\rho)\,\mathrm{d}\lambda = -Q_{0n}(\rho, t) \qquad (0 < \rho < 1),\\ &\int_0^{\infty} U_n(\lambda, t) J_n(\lambda\rho)\,\mathrm{d}\lambda = 0 \qquad (\rho > 1), \end{aligned}\right\} \tag{A 7}$$

the solution of which was also given by Sneddon (1966) in the form

$$U_n(\lambda, t) = \sqrt{\frac{\pi\lambda}{2}}\int_0^1 \mu^{1/2}\chi_n(\mu, t) J_{n+1/2}(\mu\lambda)\,\mathrm{d}\mu, \tag{A 8}$$

$$\chi_n(\mu, t) = -\frac{2\mu}{\pi}\int_0^{\pi/2} (\sin\beta)^{n+1} Q_{0n}(\mu\sin\beta, t)\,\mathrm{d}\beta. \tag{A 9}$$

Equations (A 6) and (A 8) finally give for $n \geqslant 0$

$$\psi_{1n}(\rho, 0, t) = \int_1^{1/\rho} \frac{\chi_n(\rho\nu, t)\,\mathrm{d}\nu}{\nu^n\sqrt{\nu^2 - 1}} \qquad (0 < \rho < 1), \tag{A 10}$$

which is similar to (A 5) when $n = 0$.

Appendix B. Laplacian of the zeroth-order solution

The zeroth-order solution is given by equation (52). The Laplacian of this quantity follows from equation (31). After some algebra, we obtain

$$\begin{aligned} S_0(\rho, t) = {} & -\frac{2m_1(t)}{\pi\sqrt{1-\rho^2}} + \frac{2}{\pi}m_2(t)\sqrt{1-\rho^2} + \frac{4}{\pi}\kappa_1 z_1 a^2(t) D_{-1}(\rho) \\ & -\frac{2}{\pi}\sum_{k=3}^{\infty} k(k+1)(k-2)\kappa_k z_k a^{k+1}(t) D_{k-2}(\rho), \end{aligned} \tag{B 1}$$

where

$$m_1(t) = \sum_{k=1}^{\infty} k\kappa_k z_k a^{k+1}(t) = a^2\dot{h}/\dot{a}, \qquad m_2(t) = \sum_{k=1}^{\infty} k(k+1)\kappa_k z_k a^{k+1}(t) = a\dot{m}_1/\dot{a}, \quad \text{(B 2)}$$

by using identities (51). It is easy to check that $\int_{|\rho|<1} \rho S_0(\rho, t)\, \mathrm{d}\rho \equiv 0$.

Appendix C. Calculation of the function $Q_0(\rho, \alpha, t)$

In order to obtain formula (55), the following developments are performed

$$\mathrm{Re}(T_0(\omega, \omega_0, t)) = \sum_{n=1}^{\infty} \mathrm{Re}\left[a_n(t)\frac{\omega^n - \omega_0^n}{\omega - \omega_0}\right] = \mathrm{Re}\, a_1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n-1} \mathrm{Re}\left[a_n(t)\omega^{n-m-1}\omega_0^m\right]$$

$$= \mathrm{Re}\, a_1 + \sum_{n=1}^{\infty} \mathrm{Re}\left[a_{n+1}(t)\sum_{m=0}^{n} \rho^{n-m}\rho_0^m \mathrm{e}^{\mathrm{i}[n\alpha + m(\alpha_0 - \alpha)]}\right]. \quad \text{(C 1)}$$

Equations (54)–(56) and (C 1) make it possible to rewrite equation (36) in the form

$$Q_0(\rho, \alpha, t) = \mathrm{Re}(a_1)a[h(t) - f_0(a\rho) - a\rho f_0'(a\rho)] - aA_0(a\rho)$$

$$- a^2 f_0'(a\rho)\sum_{n=1}^{\infty} \rho^{n+1}\mathrm{Re}[a_{n+1}\mathrm{e}^{\mathrm{i}n\alpha}] - a\mathrm{Re}\sum_{n=1}^{\infty} A_n(a\rho)\mathrm{e}^{\mathrm{i}n\alpha}$$

$$+ \frac{1}{2\pi}\mathrm{Re}\sum_{n=1}^{\infty}\left[a_{n+1}\sum_{m=0}^{n}\rho^{n-m}\int_0^1 S_0(\rho_0, t)\rho_0^{m+1}\left(\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}(n\alpha + m\alpha_0)}\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}}\right)\mathrm{d}\rho_0. \right. \quad \text{(C 2)}$$

The integrals with respect to α_0 in (C 2) are evaluated using standard formulæ and the following identity as well

$$\int_0^{2\pi}\frac{\sin(m\alpha_0)\,\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}} = 0. \quad \text{(C 3)}$$

Thus,

$$\int_0^{2\pi}\frac{\cos(n\alpha + m\alpha_0)\,\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}} = \cos(n\alpha)\int_0^{2\pi}\frac{\cos(m\alpha_0)\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}}, \quad \text{(C 4)}$$

$$\int_0^{2\pi}\frac{\sin(n\alpha + m\alpha_0)\,\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}} = \sin(n\alpha)\int_0^{2\pi}\frac{\cos(m\alpha_0)\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}}. \quad \text{(C 5)}$$

The integral on the right–hand side of equations (C 4) and (C 5) is known as Copson's integral (see Sneddon 1966)

$$\int_0^{2\pi}\frac{\cos(m\alpha_0)\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}} = \frac{4}{\rho^m\rho_0^m}\int_0^{min(\rho,\rho_0)}\frac{v^{2m}\,\mathrm{d}v}{\sqrt{(\rho^2 - v^2)(\rho_0^2 - v^2)}}. \quad \text{(C 6)}$$

It is convenient to introduce the new unknown function

$$M_n(\rho, t) = \frac{1}{2\pi}\sum_{m=0}^{n}\rho^{n-m}\int_0^1 S_0(\rho_0, t)\rho_0^{m+1}\left(\int_0^{2\pi}\frac{\cos(m\alpha_0)\,\mathrm{d}\alpha_0}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha_0}}\right)\mathrm{d}\rho_0. \quad \text{(C 7)}$$

Substituting (C 6) into (C 7) and changing the order of integration, we obtain

$$M_n(\rho,t)=\frac{2}{\pi}\sum_{m=0}^{n}\rho^{n-2m}\int_0^1 S_0(\rho_0,t)\rho_0\left(\int_0^{min(\rho,\rho_0)}\frac{v^{2m}\,\mathrm{d}v}{\sqrt{(\rho^2-v^2)(\rho_0^2-v^2)}}\right)\mathrm{d}\rho_0$$

$$=\frac{2}{\pi}\sum_{m=0}^{n}\rho^{n-2m}\int_0^{\rho}\frac{v^{2m}}{\sqrt{\rho^2-v^2}}\left(\int_v^1\frac{\rho_0 S_0(\rho_0,t)\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-v^2}}\right)\mathrm{d}v. \qquad \text{(C 8)}$$

The integrals in (C 8) are evaluated using formula (B 1) for the function $S_0(\rho,t)$. The following integrals are helpful for proceeding with the calculations ($n\neq 2$, $N\geqslant 0$)

$$\int_v^1\frac{\rho_0 D_{n-2}(\rho_0)\,\mathrm{d}\rho_0}{\sqrt{\rho_0^2-v^2}}=\frac{\pi}{4}\left(\frac{2v^n}{n(n-2)}-\frac{v^2}{n-2}+\frac{1}{n}\right),\quad \int_0^{\rho}\frac{v^N\mathrm{d}v}{\sqrt{\rho^2-v^2}}=\rho^N z_{N-1}, \qquad \text{(C 9)}$$

$$\int_v^1\frac{\rho_0\mathrm{d}\rho_0}{\sqrt{1-\rho_0^2}\sqrt{\rho_0^2-v^2}}=\frac{\pi}{2},\qquad \int_v^1\frac{\rho_0\sqrt{1-\rho_0^2}\mathrm{d}\rho_0}{\sqrt{\rho_0^2-v^2}}=\frac{\pi}{4}(1-v^2). \qquad \text{(C 10)}$$

Appendix D. Formulæ for a_n

Substituting (58) into (70), we arrive at the formula

$$a_{n+1}(t)=a(t)\frac{\displaystyle\int_0^{\pi/2}(\sin\beta)^{n+1}A_n(a\sin\beta)\,\mathrm{d}\beta}{\displaystyle\int_0^{\pi/2}(\sin\beta)^{n+1}q_{0n}(\sin\beta,t)\,\mathrm{d}\beta}\qquad (n\geqslant 0) \qquad \text{(D 1)}$$

for the complex coefficients $a_n(t)$ in expansion (53). Formulae (D 1) show that $a_{m+1}\equiv 0$ as soon as the corresponding coefficient $A_m(r)$ in expansion (56) is zero. Another feature, which highly simplifies the following developments, is that the denominator in (D 1) does not depend on n. Equations (51), (60) and (61) yield

$$\int_0^{\pi/2}(\sin\beta)^{n+1}q_{0n}(\sin\beta,t)\,\mathrm{d}\beta=\sum_{k=1}^{\infty}\kappa_k a^{k+1}S(n,k), \qquad \text{(D 2)}$$

where

$$S(n,k)=\frac{2}{\pi}z_k\hat{z}_0^n z_{2n}-z_{2n+k}\left(k+\frac{2}{\pi}(k+1)z_k\hat{z}_k^n\right). \qquad \text{(D 3)}$$

By using a recursive scheme, the equality

$$S(n,k)=S(0,k)=-kz_k \qquad \text{(D 4)}$$

can be proved. As a matter of fact, by definition of integrals (49)

$$z_k=\int_0^{\pi/2}\sin^{k+1}\beta\mathrm{d}\beta=\frac{\sqrt{\pi}}{2}\frac{\Gamma(k/2+1)}{\Gamma(k/2+3/2)}. \qquad \text{(D 5)}$$

Well-known properties of the gamma-function yield the recursive formula

$$z_k=\frac{\pi}{2}\frac{1}{k+1}\frac{1}{z_{k-1}},\quad z_{-1}=\frac{\pi}{2}. \qquad \text{(D 6)}$$

By using formula (D 6) twice, we find

$$z_{k+1}=\frac{k+1}{k+2}z_{k-1}. \qquad \text{(D 7)}$$

Definition (61) and equation (D 7) make it possible to prove that the product $y_n = z_{2n}\hat{z}_0^n$ satisfies the equality

$$y_{n+1} - y_n = \frac{\pi/2 - y_n}{2n+3}, \quad y_0 = \pi/2, \tag{D 8}$$

which yields $y_n = \pi/2$ for $n \geqslant 0$.

In a similar way, we introduce the quantity

$$p_n = \frac{2}{\pi} z_k \hat{z}_k^n z_{2n+k} + \frac{k}{k+1} z_{2n+k} - z_k, \tag{D 9}$$

and show, with the help of equations (D 6) and (D 7), that

$$p_{n+1} - p_n = -\frac{p_n}{2n+k+3}, \quad p_0 = 0. \tag{D 10}$$

Equation (D 10) gives $p_n = 0$ for $n \geqslant 0$ and any k.

Taking into account the obtained equations

$$z_{2n}\hat{z}_0^n = \tfrac{1}{2}\pi, \tag{D 11}$$

$$z_{2n+k}\left(k + \frac{2}{\pi}(k+1) z_k \hat{z}_k^n\right) = (k+1) z_k, \tag{D 12}$$

we can readily prove equality (D 4). Therefore, equation (D 2) takes the form

$$\int_0^{\pi/2} (\sin\beta)^{n+1} q_{0n}(\sin\beta, t)\,\mathrm{d}\beta = -\sum_{k=1}^{\infty} k z_k \kappa_k a^{k+1} = -\frac{a^2 \dot{h}}{\dot{a}}, \tag{D 13}$$

where the first equality of (B 2) has been used.

Appendix E. Instability of the axisymmetric solution

The obtained results in § 5 can be used to study the stability of the axisymmetric solution of the Wagner problem. In the stability analysis, we consider infinitesimal perturbations of the basic shape, $z = f_0(r) - h(t)$, of the entering body, which are described by the second term, $\epsilon F(r, \theta)$, in equation (1) with $\epsilon \ll 1$. We are concerned with the influence of these perturbations on the liquid flow and the shape of the contact line Γ. If the corresponding perturbations of the liquid flow and the contact line shape are also infinitesimal, we say that the axisymmetric solution of the impact problem is stable with respect to any small variations of the entering body shape. If, however, we can find a perturbation function $F(r, \theta)$, which satisfies the conditions mentioned in §1 and is such that the relative difference between the axisymmetric solution ($\epsilon = 0$) and the perturbed solution ($\epsilon > 0$) does not tend to zero as $\epsilon \to 0$, then we say that the axisymmetric solution of the Wagner problem for the function $f_0(r)$ is unstable. It is important to notice that within the stability analysis, the function $F(r, \theta)$ is not given in advance and describes small imperfections of the body shape. We do not consider here the stability of the solution with respect to small variations of the entry velocity $U(t)$.

Decompositions (56) and (72) are quite general. However, they do not cover all possible perturbations of the entering body shape. In particular, non-smooth perturbations of the body shape are not included. Asymptotic expansions (22) show that an axisymmetric solution is stable if and only if the first-order solution is bounded during the initial stage, when the basic assumptions of the Wagner theory are valid.

During the initial stage, the penetration depth $h(t)$ is much smaller than the dimension of the contact region, which is of the order of $a(t)$. Therefore, $h(t)/a(t) \ll 1$ at the stage under consideration and, correspondingly, $\dot{h}(t)/\dot{a}(t) \ll 1$. Equations (71) and (75) demonstrate that the first-order solution is unbounded if and only if $\dot{h}(t)/\dot{a}(t) \to 0$ at some time instant t. This happens, in particular, when $t \to 0$ and $\kappa_1 = 0$ in (50). If $\kappa_1 > 0$, then $\dot{h}(t)/\dot{a}(t) \to \kappa_1 z_1$ as $t \to 0$, which follows from (51). In the case $\kappa_n = 0$, $1 \leqslant n \leqslant m-1$, the perturbations which lead to instability, are those with $A_{nk} \neq 0$ in (72), where $k < m$. Indeed, in this case $a(t) = O(h^{1/m})$ as $t \to 0$ and $\dot{a}a^k/(a\dot{h}) = O(h^{k/m-1})$ in (73). It can be seen that $\dot{a}a^k/(a\dot{h}) \to \infty$ as $t \to 0$ and $k < m$. For example, if $f_0(r) = L(r/r_0)^3$, then the infinitesimal perturbation $\epsilon r^2 \cos(2\theta)/(2R)$, $\epsilon \ll 1$, of the body shape leads to a relatively large difference between the perturbed solution and the axisymmetric solution at the very beginning of the impact. On the other hand, the axisymmetric solution for the shape $f_0(r) = L(r/r_0)^3$ is stable with respect to the perturbations $\epsilon L(r/r_0)^k \cos(m\theta)$ for any m and $k > 3$. We may conclude that axisymmetric solutions for blunt shapes are unstable at the very beginning of the impact. However, before drawing any conclusion from this stability analysis, it should be stressed that several other physical phenomena, which are of importance at $t = 0^+$, were not taken into account.

Other axisymmetric shapes, which provide unstable solutions, are those for which $\dot{a} \to \infty$ well after the impact instant. Such shapes were studied by Korobkin (1995) for the two-dimensional case. In the axisymmetric impact problem, the velocity of the contact region expansion, $\dot{a}(t)$, can be calculated using (51) as

$$\dot{a}/\dot{h} = \left[\sum_{n=1}^{\infty} n\kappa_n z_n a^{n-1}(t)\right]^{-1}. \tag{E 1}$$

The velocity $\dot{a}(t)$ is beyond all bounds at a time instant t_*, when the sum in (E 1) is equal to zero. Therefore, the corresponding shapes can be recognized once the coefficients in (50) are given. It can be shown that the derivative $\dot{a}(t)$ is bounded for $t > 0$ if $f_0'(r) \geqslant 0$.

Appendix F. Asymptotics of the contact line shape

In the case of an elliptic paraboloid, $f(x, y) = x^2/(2r_x) + y^2/(2r_y)$, entering the liquid half–space at constant velocity U_0, the contact line $\Gamma(t)$ is elliptic (see Scolan & Korobkin, 2001)

$$\frac{x^2}{a_\Gamma^2(t)} + \frac{y^2}{b_\Gamma^2(t)} = 1, \tag{F 1}$$

where $a_\Gamma(t) = b_0 k_\Gamma \sqrt{t}$, $b_\Gamma(t) = b_0 \sqrt{t}$ and

$$b_0 = (2r_y U_0)^{1/2}\left[1 + k_\Gamma^2 \frac{D(e_\Gamma)}{E(e_\Gamma)}\right]^{1/2}, \tag{F 2}$$

$E(e)$ and $D(e)$ are elliptic integrals of the first and third kind, respectively. The eccentricity of the contact line, $e_\Gamma = \sqrt{1 - k_\Gamma^2}$, is determined by the equation

$$k_\Gamma^2 \frac{1 + k_\Gamma^2 D(e_\Gamma)/E(e_\Gamma)}{2 - k_\Gamma^2 D(e_\Gamma)/E(e_\Gamma)} = k_\gamma^2, \tag{F 3}$$

where $k_\gamma = r_x/r_y$ and the eccentricity of the body cross-sections e_γ is defined as

$e_\gamma = \sqrt{1-k_\gamma^2}$. In order to obtain the asymptotics of the contact line shape (F 1) for an almost axisymmetric body, we consider the case $e_\gamma \ll 1$. It is easy to show that $e_\gamma^2 = 2\epsilon/(1+\epsilon)$, where ϵ is introduced by (98) and is small for small e_γ.

In the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, equation (F 1) takes the form

$$r = a_\Gamma(t)\left[1 - \tfrac{1}{2}e_\Gamma^2(1-\cos 2\theta)\right]^{-1/2}. \tag{F 4}$$

We shall determine the asymptotic behaviour of the right-hand side in (F 4) as $\epsilon \to 0$.

Equation (F 3) gives

$$e_\Gamma^2 = \tfrac{4}{5}e_\gamma^2 + O\left(e_\gamma^4\right) \tag{F 5}$$

as $e_\gamma \to 0$. After some manipulations we obtain from equation (F 2) the following asymptotic formula

$$a_\Gamma(t) = \sqrt{3RU_0 t}\left[1 - \tfrac{2}{5}\epsilon + O(\epsilon^2)\right]. \tag{F 6}$$

Finally, equations (F 4)–(F 6) provide asymptotics (121) of the contact line as $\epsilon \to 0$

$$r = \sqrt{3RU_0 t}\left[1 - \tfrac{2}{5}\epsilon\cos 2\theta + O(\epsilon^2)\right]. \tag{F 7}$$

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